Hereditary categories

Based on the talk by Idun Reiten (Trondheim)

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Throughout the paper k denotes algebraically closed field.

An abelian category $\mathcal{H}$ is called hereditary if the functor $\operatorname{Ext}^2_{\mathcal{H}}$ vanishes.

The category $\operatorname{coh}\mathbb{P}^1(k)$ of coherent sheaves over projective line is a hereditary abelian category k-category. This category satisfies also the following conditions:

(1) Hom and Ext^1 are finite dimensional over k;

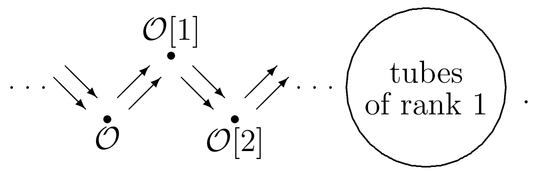
(2) the category is noetherian;

(3) we have Serre duality.

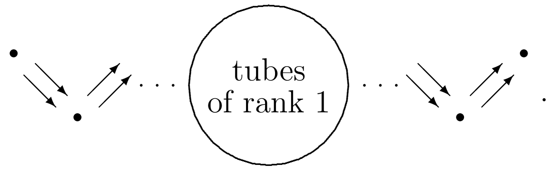
Note that $\operatorname{coh}\mathbb{P}^1(k)$ is equivalent to the quotient of the category of finitely generated graded modules over $k[X, Y]$ modulo the modules of finite length, which we will denote by $\operatorname{qgr} k[X, Y]$.

Let H be a finite dimensional hereditary k-algebra (the path algebra $k\Gamma$ of finite quiver Γ). Then $\operatorname{mod} H$, the category of finitely generated modules, is a hereditary abelian k-category satisfying (1), (2) and having almost split sequences.

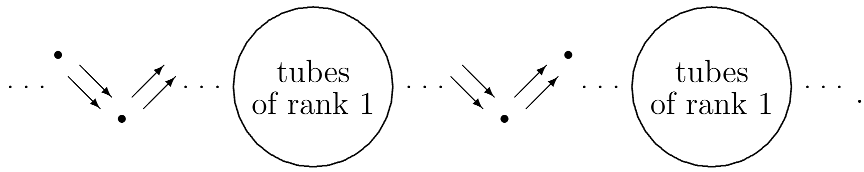
The category $\operatorname{coh}\mathbb{P}^1(k)$ has almost split sequences and the Auslander–Reiten-quiver of $\operatorname{coh}\mathbb{P}^1(k)$ is of the following from



On the other hand the Auslander–Reiten-quiver of $\operatorname{mod} k(\bullet \rightrightarrows \bullet)$ has the form



Put $T := \mathcal{O} \oplus \mathcal{O}[1]$. Then $\operatorname{End}(T)^{\mathrm{op}}$ is isomorphic to $k(\bullet \rightrightarrows \bullet)$. Note that T is a tilting object, that is $\operatorname{Ext}^1_{\mathcal{H}}(T,T) = 0$ and if $\operatorname{Hom}(T,X) = 0$ and $\operatorname{Ext}^1(T,X) = 0$ then $X = 0$. Thus we have an equivalence $D^b(\operatorname{coh}\mathbb{P}^1(k)) \sim D^b(\operatorname{mod} k(\bullet \rightrightarrows \bullet))$. Note that $D^b(\operatorname{coh}\mathbb{P}^1(k))$ is of the form



Consider $\mathcal{H} = \operatorname{coh}\mathbb{P}^1(k)$. There exists an equivalence $F : \mathcal{H} \to \mathcal{H}$ such that we have a functorial isomorphism $D\operatorname{Hom}(A,B) \simeq \operatorname{Ext}^1(B, FA)$, where D denotes the usual duality. The functor F is called Serre duality. It implies the existence of almost split sequences. Namely, we have $\operatorname{Hom}_k(\operatorname{End}(A), k) \simeq \operatorname{Ext}^1(A, FA)$. If we take A indecomposable then $\operatorname{End}(A)$ is a local ring. Hence we have the natural map $f : \operatorname{End}(A) \to k$ and we obtain an almost split sequence $0 \to FA \xrightarrow{f'} E \to A \xrightarrow{f''} 0$ via this isomorphism. The above sequences have the following properties:

(1) FA is indecomposable;

(2) the sequence is not split;

(3) for each $h : X \to A$ which is a not split epimorphism there exists an homomorphism $g : X \to E$ such that $h = f''g$.

Theorem (Reiten–Van den Bergh). *Let $\mathcal{H}$ be a hereditary abelian k-category with finite dimensional* Hom *and* Ext^1 *and with no projective nor injective objects. Then the existence of Serre duality is equivalent to the existence of almost split sequences.*

The category $\operatorname{coh}\mathbb{X}$, where $\mathbb{X}$ is the weighted projective line, is a hereditary category with the properties (1), (2) and (3).

Example. Let $R := k[X,Y,Z]/(X^2 + Y^3 + Z^5)$. This is a $\mathbb{Z}$-graded ring with $\deg X = 15$, $\deg Y = 10$ and $\deg Z = 6$. Then $\operatorname{qgr} R$ is equivalent to the the category of coherent sheaves over some weighted projective line. There exists a titling object T such that $\operatorname{End}(T)^{\mathrm{op}}$ is a canonical algebra $C(2,3,5)$. We have an equivalence $D^b(\operatorname{qgr} R) \simeq D^b(C(2,3,5))$.

Theorem (Lenzing). *Let $\mathcal{H}$ be a connected hereditary category satisfying properties* (1), (2) *and* (3) *and with no projective nor injective objects. Then $\mathcal{H}$ has a tilting object if and only if $\mathcal{H}$ is equivalent to the category* $\operatorname{coh}\mathbb{X}$ *for some weighted projective line* $\mathbb{X}$.

The algebras of the from $\mathrm{End}(T)^{\mathrm{op}}$, where T is a tilting object in a hereditary abelian k-category with finite dimensional Hom and Ext^1, are called quasi-tilted algebras.

Let R be a commutative $\mathbb{Z}$-graded Cohen–Macaulay ring of Krull dimension 2, with gradation $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ satisfying $\dim R_i < \infty$. Then we have the following.

Proposition. *The category* $\mathrm{qgr}(R)$ *is hereditary if and only if R has only isolated singularities.*

Note that if the category $\mathrm{qgr}(R)$ is hereditary then it has the properties (1), (2) and (3).

Remark. The category of graded Cohen–Macaulay modules over R has almost split sequences if and only if R has only isolated singularities. Then the category of graded Cohen–Macaulay modules is embedded into the category $\mathrm{qgr}(R)$ and into the category of all graded modules.

Idun Reiten and Michel Van den Bergh classified hereditary abelian k-categories with the properties (1), (2) and (3). One class if formed by $\mathrm{qgr}(R)$ for R as above. Another class is formed by nilpotent finite dimensional representation of $\widetilde{\mathbb{A}}_n$ with cyclic orientation and nilpotent finite dimensional representations of $\mathbb{A}_\infty^\infty$. The module categories of hereditary algebras are contained in a third larger class. We have also two more classes.