The structure of second kind modules for Galois coverings

based on the talk by Piotr Dowbor

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There are two origins of Galois coverings. First one is algebraic. Let A be a ring with grading $A = \bigoplus_{g \in G} A_g$, where G is a group. We denote by $\operatorname{mod}_G A$ the category of G-graded A-modules. We have a forgetful functor $\operatorname{mod}_G A \to \operatorname{mod} A$ and the trivial grading functor $\operatorname{mod} A \to \operatorname{mod}_G A$. We can replace investigating $\operatorname{mod}_G A$ by investigating $\operatorname{mod} \tilde{A}$ for some appropriate $\tilde{A}$.

Another source is combinatorial. Let A be the path algebra of the bounded quiver (Q, ρ). We can construct the universal cover $(\tilde{Q}, \tilde{\rho})$ of (Q, ρ). We have the action of $\Pi := \Pi(Q, \rho)$ on $(\tilde{Q}, \tilde{\rho})$ and $(Q, \rho) = (\tilde{Q}, \tilde{\rho})/\Pi$.

A k-category R is called locally bounded if

(1) $x \simeq y$ if and only if $x = y$;

(2) $R(x, x)$ is local for each $x \in R$;

(3) $\sum_{y \in R} \dim R(x, y) < \infty$ and $\sum_{y \in R} \dim R(y, x) < \infty$ for each $x \in R$.

If R is a locally bounded k-category then we denote by $\operatorname{MOD} R$ the category of R-modules, that is the category of k-linear functors from R to Vect_k. By $\operatorname{Mod} R$ we will denote the subcategory of $\operatorname{Mod} R$ formed by locally finite dimensional ones. The module M is called locally finite dimensional if for all $x \in R$ we have $\dim M(x) < \infty$. Finally, by $\operatorname{mod} R$ we denote the subcategory of $\operatorname{Mod} R$ formed by those M which are finite dimensional, that is $\sum_{x \in R} \dim M(x) < \infty$. By $\operatorname{Ind} R$ and $\operatorname{ind} R$ we will denote the subcategories of indecomposable modules in $\operatorname{Mod} R$ and $\operatorname{mod} R$, respectively. Note that if R is finite then $\operatorname{MOD} R = \operatorname{MOD} A$, where $A = A(R) := \bigoplus_{x, y \in R} R(x, y)$ is a finite dimensional algebra.

Let G be a subgroup of $\operatorname{Aut}_k(R)$. Then G acts on R and it induces the action of G on $\operatorname{MOD} R$ given by $(g, M) \mapsto {}^g M$, where ${}^g M(x) := M(g^{-1} x)$. We usually assume that G acts freely on R, that is $G_x := \{g \in G \mid gx = g\} =$

$\{e\}$ for each $x \in R$. In this case we can form the orbit category $\overline{R} := R/G$ which is again locally bounded, where $\overline{R}(\overline{x}, \overline{y}) := \prod_{x\in\overline{x}, y\in\overline{y}} R(x, y)$. We have the Galois covering $F : R \to \overline{R}$ given by $Fx = Gx$ such that for all $x, y \in R$ we have $\overline{R}(\overline{x}, Fy) = \bigoplus_{x\in\overline{x}} R(x, y)$.

Let $F^* : \mathrm{MOD}\,\overline{R} \to \mathrm{MOD}\,R$ be the functor given by $M \mapsto M \circ F^{\mathrm{op}}$ and $F_\lambda : \mathrm{MOD}\,R \to \mathrm{MOD}\,\overline{R}$ its left adjoint. Then $F_\lambda(M)(\overline{x}) = \bigoplus_{x\in\overline{x}} M(x)$, $F_\lambda(\mathrm{mod}\,R) \subset \mathrm{mod}\,\overline{R}$, and finally $F_\lambda(\mathrm{ind}\,R) \subset \mathrm{ind}\,\overline{R}$ provided G acts freely on $\mathrm{ind}\,R$, that is for each $M \in \mathrm{ind}\,R$ we have $G_M := \{g \in G \mid {}^gM \simeq M\} = \{e\}$. If G is torsion-free then G acts freely on $\mathrm{ind}\,R$.

Theorem (Gabriel)**.** *Let k be an algebraically closed field. If G is a subgroup which acts freely on* $\mathrm{ind}\,R$ *then $\overline{R}$ is representation finite if and only if R is locally representation finite. If this condition is satisfied, then F_λ induces a bijection between the G-orbits of isoclasses of indecomposable R-modules and the isoclasses of indecomposable $\overline{R}$-modules.*

There is so called Galois Covering Conjecture. Let k be an algebraically closed field and G torsion-free. Is it true that R is tame implies $\overline{R}$ is tame? The converse is always true. The Galois Covering Conjecture has been proved by Dowbor and Skowroński in so called G-exhaustive case, when F_λ is dense.

Let $\mathrm{mod}_1\,\overline{R} := \mathrm{add}\{F_\lambda M \mid M \in \mathrm{ind}\,R\}$ and $\mathrm{mod}_2\,\overline{R} := \mathrm{add}\{\mathrm{ind}\,\overline{R} \setminus \mathrm{mod}_1\,\overline{R}\}$. We call $\mathrm{mod}_2\,\overline{R}$ the category of second kind modules. Indecomposable $\overline{R}$-modules in $\mathrm{mod}_1\,\overline{R}$ can be easily characterized. Under some assumptions Dowbor and Skowroński described the category $\mathrm{mod}_2\,\overline{R}$.

We want to understand the structure of modules in $\mathrm{mod}\,\overline{R}$ and especially in $\mathrm{mod}_2\,\overline{R}$. Let $\mathrm{MOD}^G\,R$ be the category of all pairs (M, μ), where M is in $\mathrm{MOD}\,R$ and μ is an R-action of G on M, that is $\mu = \{\mu_g : M \to {}^{g^{-1}}M\}_{g\in G}$ such that $\mu_{hg} = {}^{g^{-1}}\mu_h\mu_g$. We have the functor $F_* : \mathrm{MOD}\,\overline{R} \to \mathrm{MOD}^G\,R$, $M \mapsto (M, \mu)$, where μ is the trivial action. It appears that $F_*(\mathrm{mod}\,\overline{R})$ is $\mathrm{Mod}^G_f\,R$ consisting of all (M, μ) such that $M \in \mathrm{Mod}\,R$ and $\mathrm{supp}\,M/G$ is finite.

We know that $\mathrm{End}(M)$ is local and $\mathrm{rad}\,\mathrm{End}\,M$ consists of all f such that $f(x)$ is nilpotent, for each $M \in \mathrm{Ind}\,R$. Moreover each $M \in \mathrm{Mod}\,R$ is a direct sum of $M_i \in \mathrm{Ind}\,R$. Recall that $M \in \mathrm{Mod}\,R$ is called a G-atom if M is indecomposable and $\mathrm{supp}\,M/G_M$ is finite.

Lemma. *Let $M = (M, \mu) \in \mathrm{Mod}^G_f\,R$. If $M = \bigoplus M_j$ and M_i is indecomposable then M_i is a G-atom.*

Lemma. *If M is a G-atom then G_M is finitely generated.*

Let $\mathscr{A}$ be the set of representatives of isoclasses of G-atoms and $\mathscr{A}_0 \subset \mathscr{A}$ be the set of representatives of G-orbits in $\mathscr{A}$. For $B \in \mathscr{A}_0$ we fix the set S_B of representatives of G/G_B.

Proposition. *We have that* $\operatorname{Mod}_f^G R \simeq \langle M \simeq (M_n, \mu)\rangle$, *with* $n = (n_B)_{B\in\mathscr{A}_0}$, $n_B = 0$ *for all but a finite number of* B, $M_n = \bigoplus_{B\in\mathscr{A}_0}(\bigoplus_{g\in S_B} {}^gB^{n_B})$.

We have $\operatorname{mod}\overline{R} \ni X \mapsto F_*X \simeq (M_n, \mu)$. We define $\operatorname{dss}(X) = \{B \in \mathscr{A}_0 \mid n_B \neq 0\}$ and $\operatorname{dsc}(X) = n \in \mathbb{N}^{\mathscr{A}_0}$. If $\mathscr{U} \subset \mathscr{A}_0$ then $\operatorname{mod}_{\mathscr{U}}\overline{R}$ is the category of all $X \in \operatorname{mod}\overline{R}$ such that $\operatorname{dss} X \subset \mathscr{U}$. We have $\mathscr{A} = \mathscr{A}^\infty \cup \mathscr{A}^f$ and $\operatorname{mod}_1\overline{R} = \operatorname{mod}_{\mathscr{A}^f}\overline{R}$ and $\operatorname{mod}_2 R = \operatorname{mod}_{\mathscr{A}^\infty} R$.

Tere are two problems related to Galois Covering Conjecture. First one is stablizer conjecture. It claims that if R is tame then for any $B \in \mathscr{A}^\infty$ we have $G_B \simeq \mathbb{Z}$. It was proved by Dowbor in 1999. Ii follows that special role is played by cyclic G-atoms.

Another one is connected with orbicularity of $\overline{R}$-modules. We call $X \in \operatorname{mod}\overline{R}$ orbicular if $\operatorname{dss} X$ consists of one element, that is there exists $B \in \mathscr{A}$ such that $F_*X \simeq \bigoplus_{g\in S_B} {}^gB^{n_B}$. Conjecture says that if R is tame then X is (regular) orbicular for any $X \in \operatorname{ind}\overline{R}$.