Approximations of modules, tilting theory and applications to finitistic dimension conjecture

based on the talk by Jan Trlifaj (Prague)

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Let R be an associative ring with 1. We denote by $\operatorname{Mod} R$ the category of (right) R-modules. If $\mathscr{C}$ is a class of R-modules and $M \in \operatorname{Mod} R$, then we call a map $f : C \to M$ a $\mathscr{C}$-precover of M is $C \in \mathscr{C}$ and for each $f' : C' \to M$, $C' \in \mathscr{C}$, there exists $h : C' \to C$ with $f' = fh$. We say that $\mathscr{C}$ is a precovering class if for each $M \in \operatorname{Mod} R$ there exists a $\mathscr{C}$-precover of M. The $\mathscr{C}$-precover f of M is called a $\mathscr{C}$-cover of M if f is right minimal. A class $\mathscr{C}$ is called covering if for each $M \in \operatorname{Mod} R$ there a exists $\mathscr{C}$-cover of M. Dually, we define $\mathscr{C}$-preenvelopes, $\mathscr{C}$-envelopes, preenveloping and enveloping classes.

The class of projective modules is a covering class if and only if R is a right perfect ring. Enochs showed that if R is a commutative domain then the class of torsion free modules is a covering class. There was a conjecture, called the Flat Cover Conjecture, which said that the class of flat modules is covering.

A $\mathscr{C}$-precover $f : C \to M$ of M is called a special precover if f is an epimorphism and $\operatorname{Ker} f \in \mathscr{C}^{\perp} := \operatorname{Ker} \operatorname{Ext}^1_R(\mathscr{C}, -)$. The class $\mathscr{C}$ is called special precovering if for each M there exists a special $\mathscr{C}$-precover of M. Dually one defines special preenveloping classes.

The following result is due to Wakamatsu.

Proposition. *Let $\mathscr{C}$ be closed under extensions and direct summands.*

(i) *If $\mathscr{C}$ contains all projective modules and is a covering class then $\mathscr{C}$ is a special precovering class.*

(ii) *If $\mathscr{C}$ contains all injective modules and is an enveloping class then $\mathscr{C}$ is a special preenveloping class.*

A cotorsion pair is a pair $(\mathscr{A}, \mathscr{B})$ of classes of modules such that $\mathscr{A} = {}^{\perp}\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^{\perp}$. Examples of cotorsion pairs are $(\mathscr{P}_0, \operatorname{Mod} R)$, $(\operatorname{Mod} R, \mathscr{I}_0)$ and $(\mathrm{Flat}, \mathrm{Flat}^{\perp})$, where for each n we denote by $\mathscr{P}_n$ the class of modules of projective dimension at most n and by $\mathscr{I}_n$ the class of modules of injective dimension at most n. Moreover, Flat is the class of flat modules and $\mathrm{Flat}^\perp$ is the class of so called Enochs torsion modules. If $\mathscr{C}$ is a class of modules then $({}^\perp(\mathscr{C}^\perp), \mathscr{C}^\perp)$ is called a cotorsion pair cogenerated by $\mathscr{C}$ and $({}^\perp\mathscr{C}, ({}^\perp\mathscr{C})^\perp)$ is called a cotorsion pair generated by $\mathscr{C}$.

Let $(\mathscr{A}, \mathscr{B})$ be a cotorsion pair. Then $\mathscr{A}$ is a special precovering class if and only if $\mathscr{B}$ is a special preenveloping class. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ such that $\mathscr{A}$ is a special precovering class and $\mathscr{B}$ is a special preenveloping class is called a complete cotorsion pair.

Proposition (Enochs)**.** *If $\mathscr{C}$ is a special precovering class which is closed under direct limits then $\mathscr{C}$ is a covering class. Similarly, if $\mathscr{C}$ is closed under direct limits and $\mathscr{C}^\perp$ is a special preenveloping then $\mathscr{C}^\perp$ is an enveloping class.*

Lemma. *Let $M \in \operatorname{Mod} R$ and $\mathscr{S}$ be a set of R-modules. There exists a short exact sequence $0 \to M \to A \to B \to 0$ with $A \in \mathscr{S}^\perp$ and B a $\mathscr{S}$-filtered module, i.e. $B = \bigcup_\alpha M_\alpha$ with $M_\alpha \subset M_{\alpha+1}$, $M_{\alpha+1}/M_\alpha \in \mathscr{S}$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for a limit ordinal α.*

Proof. We may assume that $\mathscr{S} = \{S\}$, since $\mathscr{S}^\perp = (\bigoplus_{X\in\mathscr{S}} X)^\perp$. Let $0 \to K \xrightarrow{\mu} F \to S \to 0$ be an exact sequence with F a free R-module. Let λ be a regular ordinal number bigger then $\operatorname{card}(K)$. We define $(P_\alpha)_{\alpha<\lambda}$ such that $P_0 = M$ and $P_\alpha \subset P_{\alpha+1}$ by induction.

For each α let $\mu_\alpha := \bigoplus_{\mathrm{Hom}_R(K,P_\alpha)} \mu \in \mathrm{Hom}(K^{(\mathrm{Hom}_R(K,P_\alpha))}, F^{(\mathrm{Hom}_R(K,P_\alpha))})$. If $\psi_\alpha : K^{(\mathrm{Hom}_R(K,P\alpha))} \to P_\alpha$ is a natural map, then for each $\eta \in \mathrm{Hom}_R(K, P_\alpha)$ there exists $\nu_\eta : K \to K^{(\mathrm{Hom}_R(K,P_\alpha))}$ such that $\psi_\alpha \nu_\eta = \eta$. Moreover, we have $\mu_\alpha \nu_\eta = \nu'_\eta \mu$ for some $\nu'_\eta : F \to F^{(\mathrm{Hom}_R(K,P_\alpha))}$. Let

$$\begin{array}{ccc} K^{(\mathrm{Hom}_R(K,P_\alpha))} & \xrightarrow{\mu_\alpha} & F^{(\mathrm{Hom}_R(K,P_\alpha))} \\ \downarrow{\scriptstyle \psi_\alpha} & & \downarrow{\scriptstyle \rho_\alpha} \\ P_\alpha & \xrightarrow{\sigma_\alpha} & P_{\alpha+1} \end{array}$$

be a pushout diagram. If α is a limit ordinal then we define $P_\alpha := \bigcup_{\beta<\alpha} P_\beta$ and we put $A := \bigcup_{\alpha<\lambda} P_\alpha$.

In order to show that $A \in S^\perp$ we need to show that each map $\tau : K \to A$ factors through F. Since $\operatorname{card}(K) < \lambda$ we have that $\operatorname{Im} \tau \subset P_\alpha$ for some α. Let $\eta : K \to P_\alpha$ be the induced map. It is enough to show that this extends to a map $\xi : F \to P_{\alpha+1}$. We have $\sigma_\alpha \eta = \sigma_\alpha \psi_\alpha \nu_\eta = \rho_\alpha \mu_\alpha \nu_\eta = \rho_\alpha v'_\alpha \mu$ thus we may take $\xi := \rho_\alpha v'_\alpha$

Since $B = A/M = \bigcup_\alpha P_\alpha/P_0$ and $(P_{\alpha+1}/P_0)/(P_\alpha/P_0) = P_{\alpha+1}/P_\alpha = \bigoplus S$, thus B is a $\mathscr{S}$-filtered module. □

Corollary. *If $\mathscr{S}$ is a set of modules then a cotorsion pair $({}^{\perp}(\mathscr{S}^{\perp}), \mathscr{S}^{\perp})$ is complete.*

Proof. For each R-module M we have a short exact sequence $0 \to M \to S \to B \to 0$ with $S \in \mathscr{S}^{\perp}$ and B a $\mathscr{S}$-filtered module. We need to show that $B \in {}^{\perp}(\mathscr{S}^{\perp})$. Let $N \in \mathscr{S}^{\perp}$. We know that $B = \bigcup_{\gamma<\delta} B_\gamma$ with $\operatorname{Ext}^1_R(B_{\gamma+1}/B_\gamma, N) = 0$ and it implies $\operatorname{Ext}^1_R(B, N) = 0$. □

We also have that, if $R \in \mathscr{S}$ then $({}^{\perp}\mathscr{S})^{\perp}$ coincides with the class of $\mathscr{S}$-filtered modules.

Corollary. *Let $\mathscr{C} = (\mathscr{A}, \mathscr{B})$ be a cotorsion pair and $M \in \operatorname{Mod}$. Then $\mathscr{C}$ is cogenerated by M if and only if $\mathscr{A}$ consists of direct summands of modules Z such that we have a short exact sequence $0 \to F \to Z \to G \to 0$ with F a free module and G an M-filtered module.*

We have the following consequences of the above results.

(1) The Flat Cover Conjecture is true. By definition the class Flat is closed under direct limits. Moreover, the cotorsion pair $(\mathrm{Flat}, \mathrm{Flat}^{\perp})$ is complete. Indeed, it is cogenerated by $\mathscr{S} := \{F \in \mathrm{Flat} \mid \operatorname{card}(F) \leq \operatorname{card}(R)\aleph_0\}$. It follows, since for any $F \in \mathrm{Flat}$ and $x \in F$, $x \neq 0$, there exists a pure submodule F' of F such that $xR \subset F'$ and $\operatorname{card}(F') \leq \operatorname{card}(R)\aleph_0$. We construct this module using the connection of pure submodules with solution sets of equations. We have that F' is flat and F/F' is flat.

(2) A module M is said to be torsion free if $\operatorname{Tor}_1^R(R/rR, M) = 0$, $r \in R$. Every module has a torsion free cover since $(\mathrm{TFree}, \mathrm{TFree}^{\perp})$ is a complete cotorsion pair and the class TFree is closed under direct limits. The class $\mathrm{TFree}^{\perp}$ is called the class of Warfield cotorsion modules.

(3) Let R be a commutative domain and Q the quotient filed of R. The pair $({}^{\perp}(Q^{\perp}), Q^{\perp})$ is a complete cotorsion pair. We call $Q^{\perp}$ the class of Matlis cotorsion modules. We have that $({}^{\perp}(Q^{\perp}), Q^{\perp}) = ({}^{\perp}(\mathscr{C}^{\perp}), \mathscr{C}^{\perp})$, where $\mathscr{C} := \operatorname{Mod} Q$ is closed under direct limits, thus $Q^{\perp}$ is an enveloping class.

(4) One can show that ${}^{\perp}\,\mathrm{PInj} = \mathrm{Flat}$, hence is a covering class, where PInj is the class of pure injective modules. In general, if $\mathscr{C}$ is a subclass of PInj then $({}^{\perp}\mathscr{C}, ({}^{\perp}\mathscr{C})^{\perp})$ is a complete cotorsion pair, ${}^{\perp}\mathscr{C}$ is a covering class and $({}^{\perp}\mathscr{C})^{\perp}$ is an enveloping class.

The answer to the question when $({}^{\perp}\mathscr{S}, ({}^{\perp}\mathscr{S})^{\perp})$ is a complete torsion pair depends on axioms of the set theory.

Let R be a ring. A module T is called a tilting module if $\operatorname{pdim} T < \infty$, $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$, $i = 1, 2, \ldots$, and there exists an exact sequence

$$0 \to R \to T_1 \to \cdots \to T_m \to 0,$$

with $T_i \in \operatorname{Add} T$, $i = 1, \ldots, m$. If $\operatorname{pdim} T \leq n$ then we say that T is an n-tilting modules.

A tilting module T is called a finite tilting module if all modules in a minimal projective resolution of T are finitely generated and there exists an exact sequence

$$0 \to R \to T_1 \to \cdots \to \cdots T_m \to 0,$$

with $T_i \in \operatorname{add} T$, $i = 1, \ldots, m$. Let T be a finite tilting module and $S := \operatorname{End}_R(T)$. It is know due to Miyashida that for each $i = 1, \ldots, n$, $n := \operatorname{pdim} T$, the functors $\operatorname{Ext}^i_R(T, -)$ and $\operatorname{Tor}^S_i(-, T)$ are quasi-inverse equivalences between $\bigcap_{\substack{j=1 \\ j\neq i}}^n \operatorname{Ker} \operatorname{Ext}^j_R(T, -)$ and $\bigcap_{\substack{j=1 \\ j\neq i}}^n \operatorname{Ker} \operatorname{Tor}^S_j(-, T)$.

Let $\mathscr{C}$ be a class of module. We denote ${}^{\perp_n}\mathscr{C} := \operatorname{Ker} \operatorname{Ext}^n_R(-, \mathscr{C})$ and ${}^{\perp_\infty}\mathscr{C} := \bigcap_n {}^{\perp_n}\mathscr{C}$

Theorem. *Let $\mathscr{C}$ be a class of modules closed under direct summands and cokernels of monomorphisms such that $\mathscr{C} \cap {}^{\perp\infty}\mathscr{C}$ is closed under direct summands. Then $\mathscr{C}$ is a special preenveloping class such that ${}^{\perp_\infty}\mathscr{C} \subset \mathscr{P}_n$ for some n if and only if there exists an n-tilting module T such that $\mathscr{C} = T^{\perp_\infty}$.*

Proof. Assume there exists an n-tilting module T such that $\mathscr{C} = T^{\perp_\infty}$. Let $\Omega^i T$ be the i-th syzygy of T. Then $\Omega^i T = 0$ for $i > n$, thus $\mathscr{C} = (\bigoplus_{i=1}^n \Omega^i T)^{\perp}$, hence is a special preenveloping class. We need to show ${}^{\perp_\infty}\mathscr{C} \subset \mathscr{P}_n$. We have ${}^{\perp_\infty}\mathscr{C} = {}^{\perp}\mathscr{C}$, since $\mathscr{C} = T^{\perp_\infty}$ is cosyzygy closed. Thus ${}^{\perp_\infty}\mathscr{C} = {}^{\perp}((\bigoplus \Omega^i T)^{\perp})$ is the class of modules filtered by $\bigoplus \Omega^i T$, which is contained in $\mathscr{P}_n$.

Assume now that $\mathscr{C}$ is a special precovering class such that ${}^{\perp_\infty}\mathscr{C} \subset \mathscr{P}_n$ for some n. Then all injective modules belong to $\mathscr{C}$, hence $\mathscr{C}$ is cosyzygy closed and ${}^{\perp}\mathscr{C} = {}^{\perp_\infty}\mathscr{C} \subset \mathscr{P}_n$. Since $\mathscr{C}$ is a special preenveloping class, we may construct sequences $0 \to C_i \to T_i \to C_{i+1} \to 0$ with $T_i \in \mathscr{C}$ and $C_{i+1} \in {}^{\perp}\mathscr{C}$, where $C_0 = R$. Since ${}^{\perp}\mathscr{C} \subset \mathscr{P}_n$, it follows that the sequence $0 \to C_{n+1} \to T_{n+1} \to C_{n+2} \to 0$ splits, thus we may assume $T_{n+1} = 0$. We put $T = \bigoplus_{i=0}^n T_i$. We have $\operatorname{pdim} T \leq n$ since $T_i \in {}^{\perp}\mathscr{C} \subset \mathscr{P}_n$. Moreover, $\operatorname{Ext}^i_R(T, T^{(I)}) = 0$, because $T \in \mathscr{C} \cap {}^{\perp}\mathscr{C}$. Thus T is a tilting module and one can show that $T^{\perp_\infty} = \mathscr{C}$. □

Corollary. *Let $\mathscr{C}$ be a torsion class. Then $\mathscr{C}$ is a special preenveloping class if and only if there exists a 1-tilting such that $\mathscr{C} = \operatorname{Gen}(T)$.*

Let $\mathscr{P} := \{M \mid \operatorname{pdim} M < \infty\}$, $\mathscr{P}^{<\infty} := \{M \in \mathscr{P} \mid \operatorname{gen} M < \infty\}$ and $\mathscr{P}_n^{<\infty} := \{M \in \mathscr{P}_n \mid \operatorname{gen} M < \infty\}$. We put $\operatorname{gldim} R := \sup_{M \in \operatorname{Mod} R} \operatorname{pdim} M$, $\operatorname{Fdim} R := \sup_{M \in \mathscr{P}} \operatorname{pdim} M$ and $\operatorname{fdim} R := \sup_{M \in \mathscr{P}^{<\infty}} \operatorname{pdim} M$. We have $\operatorname{fdim} R \leq \operatorname{Fdim} R \leq \operatorname{gldim} R$. Moreover, if $\operatorname{gldim} R < \infty$ then $\operatorname{fdim} R = \operatorname{Fdim} R = \operatorname{gldim} R$. If R is a commutative noetherian ring then $\operatorname{Fdim} R = \operatorname{Kdim} R$. In addition, if R is a local commutative noetherian ring then $\operatorname{fdim} R = \operatorname{depth} R$. Thus, if R is a local commutative noetherian ring then $\operatorname{fdim} R = \operatorname{Fdim} R$ if and only if R is a Cohen–Macaulay ring. Nagata has constructed a noetherian commutative ring such that $\operatorname{fdim} R = \infty$. There exists also a monomial finite dimensional algebra R over an algebraically closed field such that $\operatorname{fdim} R < \operatorname{Fdim} R$. It is not known, if $\operatorname{fdim} R < \infty$ for a right artin ring R.

Lemma. *Let R be a right coherent ring and $i \leq n$. There exists a tilting module T such that $(\mathscr{P}_n^{<\infty})^{\perp_i} = T^{\perp_\infty}$. We also have have* $\operatorname{pdim} T \leq n-i+1$.

Theorem. *Let R be a right noetherian ring. Then* $\operatorname{fdim} R < \infty$ *if and only if there exists a tilting module such that $(\mathscr{P}^{<\infty})^{\perp} = T^{\perp_\infty}$. In this case* $\operatorname{fdim} R \leq \operatorname{pdim} T$.

Proof. Assume $\operatorname{fdim} R < \infty$ and let $\mathscr{C} := (\mathscr{P}^{<\infty})^{\perp}$. We have $\mathscr{C} \cap {}^{\perp}\mathscr{C}$ is closed under direct sums, since if A is a finitely presented module then $\operatorname{Ext}^i_R(A, \varinjlim N_\alpha) \simeq \varinjlim \operatorname{Ext}^i_R(A, N_\alpha)$. We have also that $\mathscr{C}$ is a special preenveloping class. Moreover, ${}^{\perp_\infty}\mathscr{C} = {}^{\perp}\mathscr{C} = {}^{\perp}((\mathscr{P}^{<\infty})^{\perp}) \subset {}^{\perp}(\mathscr{P}_m^{\perp}) \subset \mathscr{P}_m$ and one can use the previous theorem.

Assume now there exists a tilting module such that $(\mathscr{P}^{<\infty})^{\perp} = T^{\perp_\infty}$. Then $\mathscr{P}^{<\infty} \subset {}^{\perp}((\mathscr{P}^{<\infty})^{\perp}) = {}^{\perp}(T^{\perp_\infty}) = {}^{\perp}((\bigoplus_{i=0}^n \Omega^i T)^{\perp}) \subset \mathscr{P}_n$. □

Theorem. *Let R be an artin algebra. Then* $\operatorname{fdim} R < \infty$ *and there exists a finitely generated tilting module T such that $(\mathscr{P}^{<\infty})^{\perp} = T^{\perp_\infty}$ if and only if $\mathscr{P}^{<\infty}$ is contravariantly finite.*

If follows that if R is an artin algebra and $\mathscr{P}^{<\infty}$ is contravariantly finite, then $\operatorname{fdim} R = \operatorname{Fdim} R < \infty$.

Let $\mathscr{A}_f := {}^{\perp}((\mathscr{P}^{<\infty})^{\perp})$ and $\mathscr{B}_f := (\mathscr{P}^{<\infty})^{\perp}$.

Theorem. *Let R be a right artin ring. Then $\mathscr{A}_f \cap \operatorname{mod} R = \mathscr{P}^{<\infty}$. Let $f_S : A_S \to S$ be a special $\mathscr{A}_f$-precover of a simple R-module S. Then* $\operatorname{fdim} R = \max_S \operatorname{pdim} A_S$.