Introduction to quantum groups and crystal bases

based on the talk by Markus Reineke (Wuppertal)

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Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. Examples of Lie algebras are $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$, $n \geq 2$. We will always assume that $\mathfrak{g}$ is a semisimple Lie algebra, i.e. $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$, where $\mathfrak{g}_i$, $i = 1, \ldots, k$, is a simple Lie algebra, that is $[-,-] \neq 0$ and for each $I \subset \mathfrak{g}_i$ such that $[\mathfrak{g}_i, I] \subset I$ we have either $I = 0$ or $I = \mathfrak{g}_i$. The semisimple Lie algebras are classified by Dynkin diagrams or, equivalently, by Cartan matrices. For example, $\mathfrak{sl}_n$ is a simple Lie algebra of type $\mathbb{A}_{n-1}$ and $\mathfrak{sl}_3$ corresponds to the matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$.

The representation of a Lie algebra $\mathfrak{g}$ in a vector space V is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. Weyl showed that if $\mathfrak{g}$ is semisimple then the category $\operatorname{mod} \mathfrak{g}$ of finite dimensional representations of $\mathfrak{g}$ is semisimple. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The categories $\operatorname{mod} \mathfrak{g}$ and $\operatorname{mod} U(\mathfrak{g})$ are equivalent, thus the category $\operatorname{mod} U(\mathfrak{g})$ is semisimple.

Recall that a complex Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. For example if $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathfrak{n}^-$ consists of the lower triangular matrices, $\mathfrak{h}$ consists of the diagonal matrices and $\mathfrak{n}^+$ consists of the upper triangular matrices. We have the generators F_i of $\mathfrak{n}^-$, H_i of $\mathfrak{h}$, E_i for $\mathfrak{n}^+$, $i \in I$, where I is the set of vertices of the corresponding Dynkin diagram. If $\mathfrak{g} = \mathfrak{sl}_n$ then $F_i = e_{i+1,i}$, $H_i = e_{i,i} - e_{i+1,i+1}$ and $E_i = e_{i,i+1}$, $i = 1, \ldots, n-1$. As the consequence $U(\mathfrak{n}^+)$ is generated by E_i, $i \in I$, as an algebra.

Let $\lambda = (\lambda_i)_{i \in I} \in \mathbb{N}I$, $I_\lambda := \sum_{i \in I} U(\mathfrak{n}^+) E_i^{\lambda_i + 1}$ and $L_\lambda := U(\mathfrak{n}^+)/I_\lambda$. We define the action of $U(\mathfrak{g})$ on L_λ via $F_i \bar{1} = 0$ and $H_i \bar{1} = \lambda_i \bar{1}$, $i \in I$. It follows that L_λ, $\lambda \in \mathbb{N}I$, form the complete set of simple $U(\mathfrak{g})$-modules.

An interesting problem connected with the above description is the question about $\dim L_\lambda$. Another one is the description of the restriction of L_λ to $U(\mathfrak{h}) = \mathbb{C}[H_i \mid i \in I]$. This is answered by Weyl character formula, which says that $\operatorname{ch} L_\lambda := \sum_{\mu \in \mathbb{N}I} \dim (L_\lambda)_\mu e^\mu = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho)}}$. However, there is still a question whether there is a "combinatorial formula" for $\operatorname{ch} L_\lambda$, i.e.

a formula of the form $(\dim L_\lambda)_\mu$ equals the number of certain combinatorial objects.

We know that $L_\lambda \otimes L_\mu = \bigoplus_{\nu \in \mathbb{N}I} L_\nu^{c^\nu_{\lambda\mu}}$ for some $c^\nu_{\lambda\mu}$. We may ask how to compute $c^\nu_{\lambda\mu}$. For type $\mathbb{A}$ the answer is contained in the Littlewood–Richardson rule.

We want to deform $U(\mathfrak{g})$. However, complex semisimple Lie algebras are rigid, that is all deformations are trivial. Consequently, $U(\mathfrak{g})$ is rigid as a cocommutative Hopf algebra. Happily, $U(\mathfrak{g})$ is not rigid as a non-cocommutative Hopf algebra. From now on we will assume that $\mathfrak{g}$ is of one of the types $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

Theorem (Serre)**.** *We have $U(\mathfrak{n}^+) = \mathbb{C}\langle E_i \mid E_i \in I\rangle/([E_i, E_j]) = 0$ if $a_{ij} = 0$, and $[E_i,[E_i,E_j]] = 0$ if $a_{ij} = -1$.*

We have $[E_i,[E_i,E_j]] = E_i^2E_j - 2E_iE_jE_i + E_jE_i^2$. Thus we may define $\mathscr{U}_v(\mathfrak{n}^+) := \mathbb{C}(v)\langle E_i \mid i \in I\rangle/([E_i, E_j] = 0$ if $a_{ij} = 0$ and $E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ if $a_{ij} = -1)$ and $U_v(\mathfrak{n}^+)$ is the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathscr{U}_v(\mathfrak{n}^+)$ generated by $E_i^{(n)}$, $i \in I$, $n \in \mathbb{N}$, where $E_i^{(n)} := \frac{1}{[n]!}E_i^n$, and $[n] := \frac{v^n - v^{-n}}{v - v^{-1}}$. It follows easily that $\mathbb{C}_1 \otimes_{\mathbb{Z}[v,v^{-1}]} U_v(\mathfrak{n}^+) \simeq U(\mathfrak{n}^+)$, where $\mathbb{C}_\mu$ denotes a 1-dimensional $\mathbb{Z}[v, v^{-1}]$-module with v acting by multiplication by μ.

Let Q be a quiver obtained from the diagram determining $\mathfrak{g}$. For $d \in \mathbb{N}I$ we define $R_d := \bigoplus_{\alpha : i \to j} \operatorname{Hom}_k(k^{d_i}, k^{d_j})$ and $G_d := \prod_{i \in I} \operatorname{GL}(k^{d_i})$, where $k = \mathbb{F}_q$ for some q. Then G_d acts on R_d via $(g_i) * (X_\alpha) := (g_j X_\alpha g_i^{-1})$. We put $\mathscr{H}(Q) := \bigoplus_{d \in \mathbb{N}I} \mathbb{C}^{G_d}(R_d)$, where $\mathbb{C}^{G_d}(R_d)$ denots the space of G_d-invariant complex functions on R_d. The formula $(f * g)(X) := q^\alpha \sum_{U \subset X} g(U)f(X/U)$ defines in $\mathscr{H}(Q)$ a structure of an associative $\mathbb{C}$-algebra called the Hall algebra. We have $\mathscr{H}(Q) \simeq \mathbb{C}_{\sqrt{q}} \otimes_{\mathbb{Z}[v,v^{-1}]} U_v(\mathfrak{n}^+)$.

Let $\mathscr{B}_q(Q)$ be the set of the characteristic functions of all orbits in all R_d. Then $\mathscr{B}_q(Q)$ is a basis of $\mathscr{H}(Q)$. There exists a basis $\mathscr{B}(Q)$ of $U_v(\mathfrak{n}^+)$, which specializes to $\mathscr{B}_q(Q)$ for each q. However, for different orientations Q of the diagram determining $\mathfrak{g}$ the bases $\mathscr{B}(Q)$ are different. Let $\mathscr{L}(Q) := \mathbb{Z}[v^{-1}]\mathscr{B}(Q)$. If follows that $\mathscr{L}(Q) = \mathscr{L}(Q')$ if Q and Q' have the same underlying graph. Thus we put $\mathscr{L} := \mathscr{L}(Q)$. If $\pi : \mathscr{L} \to \mathscr{L}/v^{-1}\mathscr{L}$ is the canonical projection, then $\pi(\mathscr{B}(Q)) = \pi(\mathscr{B}(Q'))$. We call $B := \pi(\mathscr{B}(Q))$ the crystal basis of $\mathscr{L}/v^{-1}\mathscr{L}$.

There exists the unique basis $\mathscr{B}$ of $U_v(\mathfrak{n}^+)$ such that $\mathscr{B} \subset \mathscr{L}$, $\pi(\mathscr{B}) = B$ and $\overline{b} = b$ for all $b \in \mathscr{B}$, where $\overline{E}_i = E_i$ and $\overline{v} := v^{-1}$. Th proof of the above fact uses degenerations.

Let $\mathscr{B}_\mu$ be the specialization of $\mathscr{B}$ to $\mathbb{C}_\mu \otimes_{\mathbb{Z}[v,v^{-1}]} \mathscr{B}$ and $\pi_\lambda : U(\mathfrak{n}^+) \to L_\lambda$ be the canonical projection.

Theorem (Lusztig/Kashiwara). *We have that* $\pi_\lambda(\mathscr{B}_1) \setminus \{0\}$ *is a basis of* L_λ *for all* $\lambda \in \mathbb{N}I$.

Proof. Fix $i \in I$ and choose an orientation Q such that i is a source in Q. Then, it follows that $\mathscr{B}_1(Q) \cap U(\mathfrak{n}^+)E_i^{\lambda_i+1}$ is a basis of $U(\mathfrak{n}^+)E_i^{\lambda_i+1})$. As the consequence $\mathscr{B}_1 \cap U(\mathfrak{n}^+)E_i^{\lambda_i+1}$ is a basis of $U(\mathfrak{n}^+)E_i^{\lambda_i+1}$ for all i Hence $\mathscr{B}_1 \cap I_\lambda$ is a basis of I_λ and the claim follows. □

For example we have a basis of $\mathfrak{sl}_{n+1}$, which is parameterized by triangles $(a_{ij})_{1 \leq i \leq j \leq n}$, $a_{ij} \in \mathbb{N}$. The corresponding basis of L_λ is parameterized by those (a_{ij}), which satisfy $\sum_{1 \leq k \leq i} a_{kj} - \sum_{1 \leq k < i} a_{k,j-1} \leq \lambda_j$ for $i \leq j$.