# Construction of the canonical bases

based on the talk by Markus Reineke (Wuppertal)

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Let $Q$ be a Dynkin quiver with the set of vertices $I$. Recall that $U_v(\mathfrak{n}^+) \simeq H(Q)$, where $H(Q)$ is the generic Hall algebra of $Q$. The basis of $H(Q)$ formed by $E_M$, where $M$'s are chosen representatives of the isomorphism classes of representations of $Q$. This basis corresponds to a basis $B^Q$ in $U_v(\mathfrak{n}^+)$.

**Theorem** (Lusztig)**.** *The lattice $\mathscr{L} := \bigoplus_{[M]} \mathbb{Z}[v^{-1}]E_M^Q$ does not depend on the orientation of $Q$. If $\pi : \mathscr{L} \to \mathscr{L}/v^{-1}\mathscr{L}$ is the canonical projection then $B := \pi(B_Q)$ is independent on the orientation of $Q$. Futhermore, there exists a unique basis $\mathscr{B}$ of $\mathscr{L}$ such that $\pi(\mathscr{B}) = B$ and $\overline{b} = b$ for all $b \in \mathscr{B}$, where $\overline{E_i} = E_i$ and $\overline{v} = v^{-1}$.*

The proof of the first part involves translation of BGP-reflection functors to $\mathscr{U}_v(\mathfrak{g})$, Weyl group combinatorics and explicit calculations for type $\mathbb{A}_2$. We present the proof of the second part.

Let $k$ be an algebraically closed field and $d \in \mathbb{N}I$. We define $R_d := \bigoplus_{i \to j} \operatorname{Hom}(k^{d_i}, k^{d_j})$ and $G_d := \prod_{i \in I} \operatorname{GL}(k^{d_i})$. Note that $R_d$ is an affine algebraic variety and $G_d$ is a reductive algebraic group whose action on $R_d$ is algebraic. We say $M \leq N$ if $\mathscr{O}_N \subset \overline{\mathscr{O}_M}$.

**Lemma.** *Let $M$ and $N$ be representations of $Q$. There exists a unique representation $M * N$ such that for any representation $X$ of $Q$ there exists a short exact sequence $0 \to N' \to X \to M' \to 0$ with $M \leq M'$ and $N \leq N'$ if and only if $M * N \leq X$.*

We call $M * N$ the generic extension of $M$ by $N$.

*Proof.* Let $d := \mathbf{dim}\, M$, $e := \mathbf{dim}\, N$ and $\mathscr{X}$ be the set of all elements in $R_{d+e}$ of the form $\left(\begin{smallmatrix} N' & \zeta \\ 0 & M' \end{smallmatrix}\right)$, where $N \leq N'$ and $M \leq M'$. We have a canonical projection $p : \mathscr{X} \to \overline{\mathscr{O}_M} \times \overline{\mathscr{O}_N}$, which is a trivial vector bundle. In particular, $\mathscr{X}$ is irreducible and $\mathscr{X}_0 := p^{-1}(\mathscr{O}_M \times \mathscr{O}_N)$ is a dense subset of $\mathscr{X}$.

Let $m : G_{d+e} \times \mathscr{Z} \to R_{d+e}$ be the natural map and $\mathscr{E}$ the image of $m$. Note that $X \in \mathscr{E}$ if and only if there exists a short exact sequence $0 \to N' \to X \to M' \to 0$ with $M \leq M'$ and $N \leq N'$. Moreover $\mathscr{E}_0 := m(G_{d+e} \times \mathscr{Z}_0)$ is a dense subset of $\mathscr{E}$. Since the closed subset $\mathscr{Z}$ of $R_{d+e}$ is stable under the action of the parabolic subgroup $\left\{ \left( \begin{smallmatrix} g_1 & \xi \\ 0 & g_2 \end{smallmatrix} \right) \;\middle|\; g_1 \in G_e,\ g_2 \in G_d \right\}$ of $G_{d+e}$, it follows that $\mathscr{E}$ is a closed subset of $R_{d+e}$. Thus $\mathscr{E} = \overline{\mathscr{O}_L}$ for some $L$ and the claim follows we put $M * N := L$. □

**Corollary.** *Assume* $\mathrm{Ext}^1(M, N) = 0 = \mathrm{Hom}(N, M)$*. If* $M \leq M'$ *and* $N \leq N'$ *and we have a short exact sequence* $0 \to N' \to X \to M' \to 0$ *then* $M \oplus N \leq X$*. Moreover, if* $X \simeq M \oplus N$ *then* $M' \simeq M$ *and* $N' \simeq N$*.*

*Proof.* Since $\mathrm{Ext}^1(M, N) = 0$ we trivially have $M * N = M \oplus N$ and the first part follows. To prove the second part assume that we have a short exact sequence $0 \to N' \to M \oplus N \to M' \to 0$ for some $M \leq M'$ and $N \leq N'$. Then we get $M * N' = M \oplus N$. Indeed, in general we have $M \oplus N = M * N \leq M * N' \leq M' * N'$ and the above sequence implies $M' * N' \leq M \oplus N$. Consequently, we have a short exact sequence $0 \to N' \to M \oplus N \to M \to 0$. Using that $\mathrm{Hom}(N, M) = 0$ we get $N' \simeq N$. Similarly we show $M' \simeq M$. □

In $U_v(\mathfrak{n}^+)$ we have $\overline{E_M} = \sum_{[N]} \omega_N^M E_N$ for some $\omega_N^M$. There is a problem if there is a representation theoretic interpretation of $\omega_N^M$.

**Proposition.** *If* $\omega_N^M \neq 0$ *then* $M \leq N$*. Moreover,* $\omega_M^M = 1$*.*

*Proof.* If $\dim M = 1$, then $M = E_i$ and $\overline{E_{E_i}} = E_{E_i}$.

Let $\dim M > 1$ and assume $M$ is not a power of an indecomposable representation. Then $M = M_1 \oplus M_2$, $M_1 \neq 0 \neq M_2$ and $\mathrm{Ext}^1(M_1, M_2) = 0 = \mathrm{Hom}(M_2, M_1)$. We have

$$\begin{aligned} \overline{E_M} = \overline{E_{M_1} E_{M_2}} &= \Big(\sum_{M_1 \leq A} \omega_A^{M_1} E_A\Big)\Big(\sum_{M_2 \leq B} \omega_B^{M_2} E_B\Big) \\ &= \sum_N \Big(\sum_{\substack{M_1 \leq A \\ M_2 \leq B}} \omega_A^{M_1} \omega_B^{M_2} v^{\alpha(N,A,B)} F_{AB}^N(v^2)\Big) E_N, \end{aligned}$$

thus $\omega_N^M = \big(\sum_{\substack{M_1 \leq A \\ M_2 \leq B}} \omega_A^{M_1} \omega_B^{M_2} v^{\alpha(N,A,B)} F_{AB}^N(v^2)\big)$. If $\omega_N^M \neq 0$ then there exists a short exact sequence $0 \to B \to N \to A \to 0$ with $M_1 \leq A$ and $M_2 \leq B$. Thus we get $M = M_1 \oplus M_2 \leq N$. It also follows that $\omega_M^M = 1$.

Suppose now that $M = U^a$ for an indecomposable representation $U$. Then $E_M = E_1^{d_1} \cdots E_m^{d_m} - \sum_{N \not\simeq M} v^{-\dim \mathrm{Ext}^1(N,N)} E_N$ and we can use induction. □

**Lemma.** *Let $V$ be a free $\mathbb{Z}[v, v^{-1}]$-module of finite rank with a basis $b_i$, $i \in I$ and $\bar{\cdot} : V \to V$ a $\mathbb{Z}$-linear involution such that $\overline{v} = v^{-1}$. If there exists a partial ordering on $I$ such that $\overline{b_i} = b_i + \sum_{j>i} \omega_{ij} b_j$, then there exists a unique basis $c_i$, $i \in I$, such that $\overline{c_i} = c_i$ and $c_i \in b_i + \sum_{j>i} v^{-1}\mathbb{Z}[v^{-1}] b_j$.*

If we apply the lemma to $(U_v(\mathfrak{n}^+))_d$ then we get a unique basis $\mathscr{B} = \{\mathscr{E}_M\}$ such that $\overline{\mathscr{E}_M} = \mathscr{E}_M$ and $\mathscr{E}_M = E_M + \sum_{M<N} \zeta_N^M E_n$, $\zeta_N^M \in v^{-1}\mathbb{Z}[v^{-1}]$. Lusztig Theorem now follows easily.

Note that if $M$ is a semisimple representation then $\mathscr{E}_M = E_m^{(d_m)} \cdots E_1^{(d_1)}$. Similarly, if $\mathrm{Ext}^1(M, M) = 0$ then $\mathscr{E}_M = E_1^{(d_1)} \cdots E_m^{(d_m)}$. It is also known that if $Q$ is a quiver of type $\mathbb{A}_2$ then $\mathscr{B} = \{E_1^{(a)} E_2^{(b)} E_1^{(c)} \mid b \geq a + c\} \cup \{E_2^{(a)} E_1^{(b)} E_2^{(c)} b \geq a + c\}$.