MORITA ALGEBRAS

BASED ON THE TALK BY KUNIO YAMAGA

Throughout the talk all considered algebras are k-algebras for a fixed field k.

1. Definitions

An algebra A is called selfinjective if the left A-module ${}_AA$ is injective, or, equivalently, the right A-module A_A is injective. If A is an algebra, then an A-module M is called a generator if there exists $r \in \mathbb{N}$ such that A is a direct summand of M^r. Dually, M is called a cogenerator if there exists $r \in \mathbb{N}$ such that DA is a direct summand of M^r, where $D(-) := \operatorname{Hom}_k(-, k)$. Finally, M is called faithful, if there exists $r \in \mathbb{N}$ such that A embeds into M^r.

Let A be an algebra and

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$

be a minimal injective resolution of an A-module M. We put

$$\operatorname{dom}\dim M = \sup\{n \in \mathbb{N} : I_0, \ldots, I_{n-1} \text{ are projective}\}$$

and call $\operatorname{dom}\dim M$ the dominant dimension of M. Müller has proved that $\operatorname{dom}\dim {}_AA = \operatorname{dom}\dim A_A$, and we denote this common value by $\operatorname{dom}\dim A$. If A is a selfinjective algebra, then $\operatorname{dom}\dim A = \infty$. Nakayama conjectured in 1958, that if an algebra A is not selfinjective, then $\operatorname{dom}\dim A < \infty$.

2. Morita algebras

We have the following theorem.

Theorem (Morita, 1958). *The following conditions are equivalent for an algebra A.*

(1) *There exists a selfinjective algebra B and a generator M_B such that*

$$A \simeq \operatorname{End}_B(M_B).$$

(2) *There exists a selfinjective algebra B and a generator ${}_BN$ such that*

$$A \simeq \operatorname{End}_B({}_BN)^{\mathrm{op}}.$$

Date: 18.10.2015.

(3) *There exists an idempotent $e \in A$ such that ${}_AAe$ and eA_A are faithful and injective A-modules, and*

$$A \simeq \mathrm{End}_{eAe}(Ae_{eAe}).$$

(4) *There exists an idempotent $e \in A$ such that ${}_AAe$ and eA_A are faithful and injective A-modules, and*

$$A \simeq \mathrm{End}_{eAe}({}_{eAe}eA)^{\mathrm{op}}.$$

In the situation of the theorem we call A a Morita algebra over a selfinjective algebra B. Moreover, B is called a base algebra of A.

If A a Morita algebra and A' is Morita equivalent to A, then A' is also a Morita algebra. Moreover, if B and B' are base algebras of a Morita algebra A, then B and B' are Morita equivalent. In particular, if M_B is a generator for a selfinjective algebra B and $A := \mathrm{End}_B(M_B)$, then the following conditions are equivalent:

(1) A is selfinjective;
(2) M is projective;
(3) A and B are Morita equivalent.

3. Canonical bimodules

For an algebra A we call the A-bimodule $\mathrm{Hom}_A({}_ADA, {}_AA)$ the canonical bimodule. Note that

$$\mathrm{Hom}_A({}_ADA, {}_AA) \simeq \mathrm{Hom}_A(DA_A, A_A),$$

hence we write shortly $\mathrm{Hom}_A(DA, A)$.

If A is a hereditary algebra without non-zero projective-injective modules, then $\mathrm{Hom}_A(DA, A) = 0$. On the other hand, if A is symmetric, then we have an isomorphism

$$\mathrm{Hom}_A(DA, A) \simeq A$$

of A-bimodules.

An idempotent $e \in A$ is called selfdual if we have an isomorphism

$$D(eA) \simeq Ae$$

of left A-modules, or, equivalently, we have an isomorphism

$$eA \simeq D(Ae)$$

of right A-modules. Moreover, e is called faithful if the modules ${}_AAe$ and eA_A are faithful. We have the following lemma.

Theorem. *Let e be a selfdual idempotent. Then the following hold.*

(1) *The algebra eAe is Frobenius.*
(2) *We have an isomorphism*

$$D(eA)_{\nu_A} \simeq Ae$$

of A-eAe-bimodules.

(3) *The algebra eAe is symmetric if and only if we have an isomorphism*

$$D(eA) \simeq Ae$$

of A-eAe-bimodules.

The following is the first main result of the talk.

Theorem (Kerner/Yamagata, 2013)**.** *If V is the canonical bimodule for an algebra A, then the following conditions are equivalent.*

(1) *A is Morita algebra.*
(2) *The module ${}_AV$ is faithful and* $\operatorname{dom}\dim A \geq 2$.
(3) *The module V_A is faithful and* $\operatorname{dom}\dim A \geq 2$.
(4) *The canonical map*

$$A \to \operatorname{End}_A(V_A)$$

is an isomorphism.

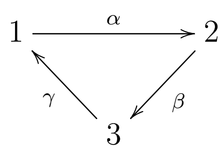
(5) *The canonical map*

$$A \to \operatorname{End}_A({}_AV)$$

is an isomorphism.

One should note that the module ${}_AV$ is faithful (equivalent, the module V_A is faithful) if and only if there exists a faithful and selfdual idempotent $e \in A$. Consequenently, A is a Morita algebra if and only if $\operatorname{dom}\dim A \geq 2$ and there exists a faithful and selfdual idempotent $e \in A$. We illustrate this observation by examples.

First, let Q be the quiver



and $A := kQ/\langle \alpha\gamma, \beta\alpha \rangle$. Then $\operatorname{dom}\dim A = 3$ and $e_1 + e_2$ is a faithful and selfdual idemponent, hence A is a Morita algebra. On the other hand, if Q is the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $A := kQ/\langle \beta\alpha \rangle$, then $\operatorname{dom}\dim A = 2$, but there is no faithful and selfdual idempotent in A.

The following is the second main result of the talk.

Theorem (Kerner/Yamagata, 2013)**.** *If V is the canonical bimodule for an algebra A, then the following conditions are equivalent.*

(1) *A is a Morita algebra.*
(2) *The module ${}_AV$ is projective.*
(3) *The module V_A is projective.*
(4) *The module ${}_AV$ is a generator.*

(5) *The module* V_A *is a generator.*
(6) *There exists a generator* M_B *for a Frobenius algebra* B *such that*

$$A \simeq \mathrm{End}_B(M_B) \qquad \textit{and} \qquad \mathrm{add}(M) = \mathrm{add}(M_{\nu_B}).$$

(7) *There exists a generator* ${}_BN$ *for a Frobenius algebra* B *such that*

$$A \simeq \mathrm{End}_B({}_BN) \qquad \textit{and} \qquad \mathrm{add}(N) = \mathrm{add}({}_{\nu_B}N).$$

We obtain the following corollaries of the above theorem.

Corollary. *If* V *is the canonical bimodule for an algebra* A*, then the following conditions are equivalent.*

(1) *The modules* ${}_AV$ *and* ${}_AA$ *are isomorphic.*
(2) *The modules* V_A *and* A_A *are isomorphic.*
(3) *There exists a generator* M_B *for a Frobenius algebra* B *such that*

$$A \simeq \mathrm{End}_B(M_B) \qquad \textit{and} \qquad M \simeq M_{\nu_B}.$$

(4) *There exists a generator* ${}_BN$ *for a Frobenius algebra* B *such that*

$$A \simeq \mathrm{End}_B({}_BN) \qquad \textit{and} \qquad N \simeq {}_{\nu_B}N.$$

Corollary (Fang/Koenig, 2011)**.** *The following conditions are equivalent for an algebra* A*.*

(1) A *is a Morita algebra over a symmetric algebra.*
(2) *There exists a faithful and selfdual idempotent* $e \in A$ *such that we have an isomorphism*

$$D(eA) \simeq Ae$$

of A*-*eAe*-bimodules.*

(3) *The* A*-bimodules* $\mathrm{Hom}_A(DA, A)$ *and* A *are isomorphic.*