A NON-LEVI BRANCHING RULE IN TERMS OF LITTELMANN PATHS

BASED ON A TALK BY JACINTA TORRES

1. The branching problem

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Denote by $\mathrm{Irr}(\mathfrak{g})$ a set of representatives of the isomorphism classes of the irreducible representations of $\mathfrak{g}$.

Theorem (Weyl). *If V is a finite dimensional representation of $\mathfrak{g}$, there there exist uniquely determined nonnegative integers m_σ such that*

$$V \simeq \bigoplus_{\sigma \in \mathrm{Irr}(\mathfrak{g})} \sigma^{m_\sigma}.$$

Note that Weyl's theorem is an analogue of Mashke's theorem from representation theory of finite groups.

Let $\mathfrak{l}$ be a semisimple Lie subalgebra of $\mathfrak{g}$. The branching problem is to find, for an irreducible representation of $\mathfrak{g}$, the decomposition of $\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}} V$. A similar problem in the case of the embedding $\mathbb{S}_m \subseteq \mathbb{S}_n$, $m \leq n$, has a well-know solution, namely

$$\mathrm{res}^{\mathbb{S}_n}_{\mathbb{S}_m} V(\lambda) = \bigoplus_{\mu} V(\mu)^{m(\lambda,\mu)},$$

where $m(\lambda,\mu)$ is the number of paths from μ to λ in the Young lattice.

2. Notation for Lie algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. We denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ is the algebra of $n \times n$ matrices with trace 0, then $\mathfrak{h}$ is the subspace of the diagonal $n \times n$ matrices with trace 0.

For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\alpha := \{g \in \mathfrak{g} : [h,g] = \alpha(h)g, \text{ for each } h \in \mathfrak{h}\}.$$

Then $\mathfrak{g}_0 = \mathfrak{h}$. By R we denote the set of $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. We call the elements of R the roots of $\mathfrak{g}$. We also denote by Λ a (chosen) set of simple roots. It is known that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

which is called the root space decomposition of $\mathfrak{g}$.

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For example, if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then

$$R = \{\varepsilon_i - \varepsilon_j : i \neq j\},$$

where $\varepsilon_i(\mathrm{diag}(a_1, \ldots, a_n)) = a_i$, and

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{i,j}.$$

Moreover, as Λ we may choose the set consisting of $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots, n-1$.

We call $\mu \in \mathfrak{h}^*_{\mathbb{R}}$ an integral dominant weight, if $\langle \mu, \alpha \rangle := \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$ is a nonnegative integer, for each $\alpha \in \Delta$. The cone spanned by the integral dominant weights is called the fundamental Weyl chamber. It is known that the irreducible representations of $\mathfrak{g}$ correspond bijectively to the integral dominant weights. If λ is an integral dominant weight, then we denote the corresponding representation by $L(\lambda)$.

It is known that if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then the integral dominant weights correspond to the partitions with at most $n-1$ part, where the bijection is given by the formula

$$(a_1, \ldots, a_{n-1}) \mapsto a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + a_{n-1}\varepsilon_{n-1}.$$

Let L be an irreducible representation of $\mathfrak{g}$. For $\mu \in \mathfrak{h}^*$, we put

$$L_\mu := \{l \in L : h \cdot l = \mu(h)l, \text{ for each } h \in \mathfrak{h}\}.$$

If weight(L) is the set of $\mu \in \mathfrak{h}^*$ such that $L_\mu \neq 0$, then

$$L = \bigoplus_{\mu \in \mathrm{weight}(L)} L_\mu$$

and we call it the weight decomposition of L.

3. Littelmann pathmodel

Let L be an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. By a Littelmann pathmodel of L we mean a set $P(L)$ of paths $[0, 1] \to \mathfrak{h}^*_{\mathbb{R}}$ such that the following conditions are satisfied:

(1) if $\mu \in$ weight(L), then the number of $\eta \in P(L)$ with $\eta(1) = \mu$ equals $\dim L_\mu$;
(2) $\eta(0) = 0$, for each $\eta \in P(L)$,
(3) $P(L)$ is generated by a single path by applying certain operations.

If $L = L(\lambda)$, then we write $P(\lambda)$ instead of $P(L(\lambda))$.

We present a construction of $P(\lambda)$ in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and λ is a partition with at most $n-1$ part. Let $\mathcal{T}(\lambda)$ be the set of Young tableaux of shape λ filled with letters 1, 2, ..., n in a nondecreasing order in each row and in a increasing order in each column. For $T \in \mathcal{T}(\lambda)$ we denote by $w(T)$ the word obtained from T be reading each column (starting from the right) from the top to the bottom. For example, if T is the diagram

$$\begin{matrix} 1 & 2 & 2 \\ 2 & 3 & \end{matrix}$$

of shape $(3, 2)$, then $w(T) = 22312$. For $i \in \{1, \ldots, n\}$, let $\pi_i \colon [0, 1] \to \mathfrak{h}_{\mathbb{R}}^*$ be given by $\pi_i(t) := t\varepsilon_i$. If $w(T) = w_1 \cdots w_s$, then

$$\pi_T := \pi_{w_1} * \cdots * \pi_{w_s}.$$

Finally,

$$P(\lambda) := \{\pi_T : T \in \mathcal{T}(\lambda)\}.$$

The following two theorems due to Littelmann are a sample of the strength of this model. First, a generalized Littlewood–Richardson rule says that

$$L(\lambda) \otimes L(\mu) = \bigoplus_{\eta} L(\eta(1)),$$

where η runs through the paths of the form $\nu * \pi$ with $\nu \in P(\mu)$ and $\eta \in P(\lambda)$, which are contained in the fundamental Weyl chamber. Next, a Levi branching rule states that if $\Delta' \subseteq \Delta$, then

$$\operatorname{res}^{\mathfrak{g}}_{\mathfrak{g}^{\Delta'}} L(\lambda) = \bigoplus_{\eta} L^{\Delta'}(\eta(1)),$$

where η runs through the paths in $P(\lambda)$, which are dominant for $\mathfrak{g}^{\Delta'}$.

4. The main result

Let $\mathfrak{g} = \mathfrak{sl}(2n, \mathbb{C})$ and σ be the automorphism of $\mathfrak{g}$ given by folding the Dyning diagram of type $\mathbb{A}_{2n-1}$. The set $\mathfrak{g}^\sigma$ of fixed points of σ is a semisimple Lie algebra isomorphic to a symplectic Lie algebra. Moreover, $\mathfrak{h}^\sigma$ is a Cartan subalgebra of $\mathfrak{g}^\sigma$ and the restrictions of the simple roots form a set of simple roots for $\mathfrak{g}^\sigma$.

For a partition λ and $\pi \in P(\lambda)$ we denote by $\operatorname{res}(\pi)$ the path given by restricting π to $\mathfrak{h}_{\mathbb{R}}^*$. Let $\operatorname{domres}(\lambda)$ be the set of such restrictions, which are contained in the Weyl fundamental chamber for $\mathfrak{g}^\sigma$ (with respect to the given set of simple roots). The main result of the talk is the following.

Theorem (Schumann, Torres)**.** *In the above situation*

$$\operatorname{res}^{\mathfrak{g}}_{\mathfrak{g}^\sigma} L(\lambda) = \bigoplus_{\lambda \in \operatorname{domres}(\lambda)} L^\sigma(\eta(1)).$$