ON THE ASYMPTOTIC GROWTH OF BIRKHOFF INTEGRALS FOR LOCALLY HAMILTONIAN FLOWS AND ERGODICITY OF THEIR EXTENSIONS

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Abstract. We consider smooth area-preserving flows (also known as *locally Hamiltonian flows*) on surfaces of genus $g \geq 1$ and study ergodic integrals of smooth observables along the flow trajectories. We show that these integrals display a *power deviation spectrum* and describe the cocycles that lead the pure power behaviour, giving a new proof of results by Forni (Annals 2002) and Bufetov (Annals 2014) and generalizing them to observables which are non-zero at fixed points. This in particular completes the proof of the original formulation of the Kontsevitch-Zorich conjecture. Our proof is based on building suitable *correction operators* for cocycles with logarithmic singularities over a full measure set of interval exchange transformations (IETs), in the spirit of Marmi-Moussa-Yoccoz work on piecewise smooth cocycles over IETs. In the case of symmetric singularities, exploiting former work of the second author (Annals 2011), we prove a tightness result for a finite codimension class of observables. We then apply the latter result to prove the existence of ergodic infinite extensions for a full measure set of locally Hamiltonian flows with non-degenerate saddles in any genus $g \geq 2$.

1. Introduction and main results

In this paper we give a contribution to the study of ergodic theory of smooth area-preserving flows on higher genus surfaces (also known as locally Hamiltonian flows) as well as to the infinite ergodic theory of flow extensions. The class of surface flows that we work with is introduced in § 1.1. We study in particular *deviations of ergodic averages*, by proving the existence of a *power deviation* spectrum for the ergodic integrals along the flow. This extends and gives a new proof of results by Forni [23] and Bufetov [6] for observables with compact support *outside* a neighbourhood of the fixed points of the flow, to observables which have full support and are *non-zero* at singularities. We then use our result to show the existence of infinite extensions of such flows which are *ergodic* with respect to the natural infinite invariant measure. This result generalizes to higher genus a classical result by Krygin [42] in genus one and extends a previous result in higher genus by the authors (see [20], where we showed the existence of ergodic extensions in any genus, but only for flows with self-similar foliations) to a full measure set of flows.

1.1. Locally Hamiltonian flows.

Let M be a compact, connected, orientable (smooth) surface and let g denote its genus. We will assume throughout that $g \geq 1$. We will consider smooth flows on M preserving a smooth measure μ (i.e. absolutely continuous measure with smooth positive density), see § 2.1. These flows, also known in the literature as *multi-valued Hamiltonian*, are *locally Hamiltonian* flows: indeed, the flow $\psi_{\mathbb{R}} := (\psi_t)_{t\in\mathbb{R}}$ is *locally* Hamiltonian in the sense that around any point in M one can find coordinates (x_1, x_2) on M in which $\psi_{\mathbb{R}}$ is locally given by the solution to the equations

$$\begin{cases} \dot{x}_1 & = \partial H/\partial x_2, \\ \dot{x}_2 & = -\partial H/\partial x_1 \end{cases}$$

for some smooth real-valued Hamiltonian function H. A *global* Hamiltonian H cannot be in general defined (see [50], § 1.3.4), but one can think of $\psi_{\mathbb{R}}$ as globally given by a *multi-valued* Hamiltonian function. We will assume throughout this paper that the fixed points of $\psi_{\mathbb{R}}$ are *non-degenerate* (also called *Morse* fixed points), namely that for every fixed point p the local Hamiltonian H is a Morse function at p.

The interest in the study of multi-valued Hamiltonians and the associated flows in higher genus ($g \geq 1$) and, in particular, in their ergodic and mixing properties, was highlighted by Novikov [51] in connection with problems arising in solid-state physics as well as in pseudo-periodic topology (see e.g. the survey [72] by A. Zorich). The simplest examples of locally Hamiltonian flows with singularities on a torus, i.e. flows with one center and one simple saddle (see Figure 1(a)), were studied by V. Arnold in [2] and are nowadays often called *Arnold flows*[1].

On the space of locally Hamiltonian flows, one can define a *topology* (see § 2.1.1) as well as a measure class (the *Katok fundamental class*, see § 2.1.2). Our understanding of the *typical* chaotic properties (in the measure theoretical sense) of these flows has advanced a lot in the last forty years. While results concerning

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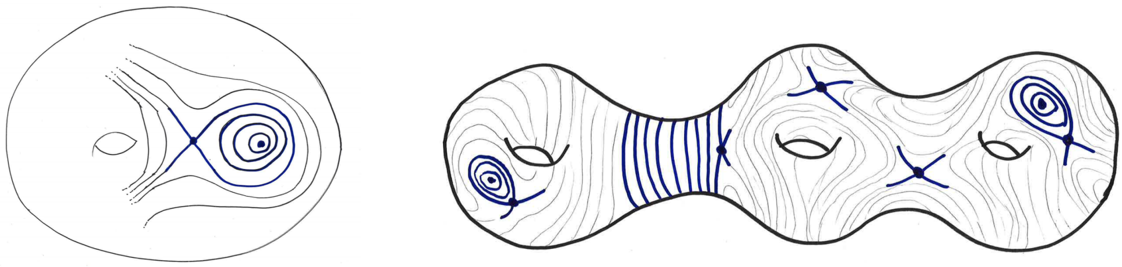
[1]More precisely, referring to the decomposition described in § 2.1.1, we call *Arnold flow* the restriction to a minimal component obtained by removing the center and the disk filled by periodic orbits around it (called *island*), which, as Arnold shows in [2], is always bounded by a saddle loop.



(a) An Arnold flow ($g = 1$) (b) A flow on a surface of $g = 3$

FIGURE 1. Examples of locally Hamiltonian flows on a surfaces.

orbit properties, such as minimality or ergodicity, were known first, since they can be simply deduced[2] from classical results which were proved using Teichmüller dynamics (see below as well as § 2.1.4), results on finer chaotic properties such as (weak or multiple) mixing, or recently spectral and disjointness results, were proved only in the last twenty years, since they depend on the movement along trajectories (i.e. on time-reparametrization) and require more delicate estimates exploiting the locally Hamiltonian parametrization of the orbits. We summarize some of the known results in § 2.1.4 below.

In the classification of chaotic behaviour in locally Hamiltonian flows it is crucial to distinguish between two open sets (complementary, up to measure zero, see § 2.1.2 for more details): in the first open set, which we will denote by $\mathscr{U}_{min}$, the typical flow is *minimal*, in the sense that the orbits of all points which are not fixed points are *dense* in M. On the other open set, that we call $\mathscr{U}_{\neg min}$, the flow is not minimal, but one can decompose the surface into a finite number of subsurfaces with boundary M_i, $i = 1, \dots, N$ such that for each i either M_i is a *periodic component*, i.e. the interior of M_i is foliated into closed orbits of $\psi_{\mathbb{R}}$ (in Figure 1 (b) one can see three periodic components, namely two disks and one cylinder, all foliated by closed orbits), or M_i is such that the restriction of $\psi_{\mathbb{R}}$ to M_i is minimal in the sense above (two such subsurfaces are visible in the example in Figure 1 (b)). The latter are called *minimal components* and there are at most g of them (where g is the genus of M), see § 2.1.4.

The study of locally Hamiltonian flows is intertwined with the study another famous class of flows on surfaces, namely *translation* (*linear*) *flows*[3] on translation surfaces, which are at the center of Teichmüller dynamics. Each minimal component of a locally Hamiltonian flow $\psi_{\mathbb{R}}$ indeed can be seen as a *time-reparametrization* (or a *time-change*) of a translation flow. Notice though that the time-change is *singular* at the fixed points $\mathrm{Fix}(\psi_{\mathbb{R}})$ of $\psi_{\mathbb{R}}$ (see § 2.3.2 and Remark 2.3 for a more precise description of the relation). One of the results which can be inferred from classical results on translation flows (proved through Teichmüller dynamics) is that the *typical* flow (in the measure theoretical sense) in $\mathscr{U}_{min}$ is *ergodic* (with respect to μ) and the typical flow in $\mathscr{U}_{\neg min}$ is ergodic when restricted to each minimal component (see § 2.1.4); it also follows that the associated foliation into flow trajectories (or equivalently any Poincaré map of the flow) is *uniquely* ergodic (i.e. there is an unique invariant probability transverse measure, the transverse measure induced by μ). Notice, though, that any locally Hamiltonian flow $\psi_{\mathbb{R}}$ with $Fix(\psi_{\mathbb{R}}) \neq \emptyset$ is *not* uniquely ergodic (as a smooth flow on a compact manifold): indeed, in the presence of singularities, there are always trivial invariant measures (Dirac deltas) supported at singularities. The presence of such measures and their effect on ergodic integrals plays a key role in this work.

1.2. **Power deviations and asymptotic behaviour of ergodic averages.** Let $\psi_{\mathbb{R}}$ denote either a locally Hamiltonian flow on M in $\mathscr{U}_{min}$ or the restriction of $\psi_{\mathbb{R}}$ in $\mathscr{U}_{\neg min}$ to a minimal component M_i, that by abusing the notation we will again denote by M here, and assume that $\psi_{\mathbb{R}}$ is ergodic (and the associated foliation is uniquely ergodic). Thus, for every smooth observable $f : M \to \mathbb{R}$ and for *almost every*[4] initial point $p \in M$,

[2]One can show (see for example [72]) that every *minimal* locally Hamiltonian flow on M (as well as the restriction of a locally Hamiltonian flow to one of its minimal components (see § 2.1.1) has the same *trajectories* (up to time-reparametrization) as a *translation flow*. Thus, one can infer properties which depend only on trajectories as sets and not on their time-parametrization, such as minimality and ergodicity, from the known properties of typical translation flows.

[3]Translation flows are unit speed *linear flows* on *translation surfaces*, namely surfaces which are locally Euclidean outside a finite number of conical singularities with cone angles of angle $2\pi k, k \in \mathbb{N}$. On these surfaces, one has a well defined notion of direction and for each $\theta \in S^1$ one can define a directional flow $h^\theta_{\mathbb{R}} = (h^\theta_t)_{t\in\mathbb{R}}$ which moves points along lines in direction θ at unit speed.

[4]Equidistribution of almost very point follows simply by ergodicity and Birkhoff ergodic theorem. *Unique* ergodicity yields a stronger conclusion *if* the observable if supported *outside* the set of fixed points $Fix(\psi_{\mathbb{R}})$: in this case equidistribution, namely (1.1), holds for any *regular* p, i.e. any p such that its forward orbit is $(\psi_t(p))_{t\geq 0}$ is dense). One can show though, that this is not the case for observables f which are non-zero at some fixed points, namely there are regular points for which equidistribution does not hold.

the ergodic averages of f converge to the spatial averages, i.e.

$$\lim_{T\to+\infty}\frac{I_T(f,p)}{T}=\int_M f\,\mathrm{d}\,\mu,\ \text{ where } I_T(f,p)=I_T(f,p,\psi_{\mathbb{R}}):=\int_0^T f(\psi_t(p))\mathrm{d}\,t. \tag{1.1}$$

With *deviations of ergodic averages* one refers to the study of the *oscillations* of the ergodic integrals $I_T(f,p)$ (or the related Birkhoff sum over an interval exchange map obtained as Poincaré section) of an observable $f:M\to\mathbb{R}$ of zero mean $\int_M f(p)\,\mathrm{d}\mu=0$ over the orbit of (typical) point $p\in M$. A distinctive phenomenon first discovered experimentally by A. Zorich in the 1990s (see [70] and also [40, 70]) is that deviations of ergodic averages have *polynomial nature*, in the following sense: for a typical flow, for suitable classes of observables, one can find an exponent $\nu=\nu(f)$ with $0<\nu<1$ such that, $I_T(f,p)\sim O(T^\nu)$ for every regular point p, where we use the notation

$$I_T(f,p)\sim O(T^\nu)\qquad\Leftrightarrow\qquad \limsup_{T\to\infty}\frac{\log I_T(f,p)}{\log T}=\nu. \tag{1.2}$$

Kontsevich and Zorich explained this phenomenon heuristically using renormalization and conjectured that, at least in the case of locally Hamiltonian flows with non-degenerate fixed points[5], there is a full *deviation spectrum*, namely there are exactly g positive exponents $0<\nu_g<\cdots<\nu_2<\nu_1:=1$ and a corresponding filtration of $H_{g+1}\subset H_g\subset\cdots\subset H_1$ of the space of smooth functions such that if $f\in H_i\backslash H_{i+1}$, with $1\le i\le g$, then $I_T(f,p)\sim O(T^{\nu_i})$ (see [40]). Zorich gave in [71] a rigorous proof of this phenomenon for ergodic integrals of a special class of functions $f:M\to\mathbb{R}$, those which represent cohomology classes[6]. Forni proved most of this conjecture in [23] (with the exception of *simplicity*, namely the strict inequalities between $\nu_g<\nu_{g-1}<\cdots<\nu_1$, which was later proved by Avila and Viana in [5], while the positivity of $\nu_g>0$ is a crucial part of [23]) for smooth observables and typical flows in the closely related class of translation flows on translation surfaces (see footnote 3). In the setting of locally Hamiltonian flows, he considers the minimal case $\psi_{\mathbb{R}}\in\mathcal{U}_{min}$ and has the further assumption that the (smooth) observable f is *compactly supported* outside of a neighbourhood of the finite set of fixed points $\mathrm{Fix}(\psi_{\mathbb{R}})$ (or, more generally, in the Sobolev regularity setting, that at least the function f vanishes on $\mathrm{Fix}(\psi_{\mathbb{R}})$, see [23] as well as [25]). We comment below on the consequences of this assumption (see Remark 1.1).

The *power spectrum* of ergodic integrals is related in [71, 23] to *Lyapunov exponents* of the Kontsevich-Zorich cocycle (so that in particular the strict inequalities $\nu_g<\nu_{g-1}<\cdots<\nu_1$ hold in view of the simplicity of the Lyapunov spectrum, which is the result later shown by Avila-Viana in [5] work); the filtration is described by Forni in [23] in terms of kernels of what we nowadays call *Forni's invariant distributions.* We refer the interested reader to [72, 24, 25, 5] for surveys of this phenomenon; in [24] other instances of *parabolic* flows for which deviations can be studied via *renomalization* are also mentioned.

A finer analysis of the behaviour of Birkhoff sums or integrals, beyond the *size* of oscillations, appears in the work [6] by Bufetov, as well as in the work [45] by Marmi, Moussa and Yoccoz. In [6], Bufetov studies limit theorems for ergodic integrals of translation flows (and describe weak limit distributions) in terms of objects that he calls *Hölder cocycles* (or, in the more general context of Markov compacta, *finitely-additive measures*) and turn out to be *dual* to Forni's invariant distributions (see [6] for details). In particular, he shows that for a full measure set of translation flows $h_{\mathbb{R}}:=(h_t)_{t\in\mathbb{R}}$ (with respect to the Masur-Veech measure), there exists $g-1$ cocycles[7] $\Phi_i(t,x):\mathbb{R}\times M\to\mathbb{R}$, for $i=2,\dots,g$ (closely related to the *limit shapes* introduced independently at the same time by Marmi, Moussa and Yoccoz in [45]), each of which has a *pure power* growth, i.e. such that $|\Phi_i(T,x)|\sim O(T^{\nu_i})$ (in the sense of (1.2) above), which, together with the trivial cocycle $\Phi_1(t,x)=t$, encode the *asymptotic behaviour* of the ergodic integrals along the flow, by providing an *asymptotic expansion* up to subpolynomial terms, i.e. such that

$$I_T(f,p,h_{\mathbb{R}})=\int_0^T f(h_s(p))\mathrm{d}s=c_1T+c_2\Phi_2(T,p)+\cdots+c_g\Phi_g(T,p)+err(f,T,p),$$

where the *error term* $err(f,T,p)$ is subpolynomial, i.e. for any $\epsilon>0$ there exists $C_\epsilon>0$ such that $|err(f,T,p)|\le C_\epsilon T^\epsilon$. The constant of the linear leading term is $c_1=\int f\mathrm{d}\omega$, where ω is the underlying translation surface area form, and the other coefficients can be computed evaluating invariant distributions D_i for $i=1,\dots,g$, i.e. $c_i=D_i(f)$.

[5] This is the framework proposed in the paper [40], where Kontsevich (based on joint work with Zorich) formulates the conjecture on the existence of the deviation spectrum (which later became known as *Kontsevich-Zorich conjecture*). They first state the result for homology classes (or equivalently characteristic functions over interval exchange transformations) and then suggest that the phenomenon should hold more generally if one considers, for simplicity, locally Hamiltonian flows with Morse saddles and the space of smooth functions.

[6] In the setting of [71], this class of functions reduces to the study of Birkhoff sums of piecewise constant functions over interval exchange maps.

[7] Here $\Phi_i(t,x)$ is a *cocycle* over the flow $h_{\mathbb{R}}$ in the sense that $\Phi_i(t+s,x)=\Phi_i(t,x)+\Phi_i(s,h_t(x))$

1.3. **Ergodicity of extensions.** A classical way to visualize and study the behaviour of ergodic averages of an observable $f : M \to \mathbb{R}$ along the flow $\psi_{\mathbb{R}}$ on M is to consider the flow on $M \times \mathbb{R}$ given by coupling $\psi_{\mathbb{R}}$ with the differential equation on $\mathbb{R}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(\psi_t(p)), \qquad y \in \mathbb{R}, \quad t \in \mathbb{R}.$$

One can see that the solution is given by the flow $\Phi^f_{\mathbb{R}} := (\Phi^f_t)_{t\in\mathbb{R}}$ on $M \times \mathbb{R}$ given by the formula

$$\Phi^f_t(p, y) = \left(\psi_t(p),\ y + \int_0^t f(\psi_s(p))\,\mathrm{d}s\right), \qquad p \in M,\ y \in \mathbb{R},\ t \in \mathbb{R}. \tag{1.3}$$

Thus, the flow $\Phi^f_{\mathbb{R}}$ is a *skew product* and provides an *extension* to $M \times \mathbb{R}$ of the flow $\psi_{\mathbb{R}}$ on M (i.e. it *projects* on the M coordinate to the flow $\psi_{\mathbb{R}}$). The motion in the $\mathbb{R}$ fiber is determined by the oscillations of the *ergodic integrals* of f along $\psi_{\mathbb{R}}$. Notice that $\Phi^f_{\mathbb{R}}$ preserves the *infinite* product measure $\mu \times Leb$, where μ is the invariant measure for $\psi_{\mathbb{R}}$ and Leb denotes the Lebesgue measure on $\mathbb{R}$.

The study of these type of skew products goes back to Poincaré [52] and his work on differential equations on $\mathbb{R}^3$ (in the case when $\psi_{\mathbb{R}}$ is a smooth flow on the torus); the study of infinite skew product extensions in greater generality became later a central topic in *infinite ergodic theory*, see for example the monographs [1, 59]. A basic question is whether the flow $\Phi^f_{\mathbb{R}}$ is ergodic (see § 2.1.3) or, if not, what is a description of ergodic components. A necessary condition for ergodicity is that f has zero mean, i.e. $\int_M f\mathrm{d}\mu = 0$, since otherwise $\Phi^f_{\mathbb{R}}$ has a *drift* and is not even *recurrent* (see § 2.1.3). In the setting of extensions, a property completely opposite to ergodicity is *reducibility.* If the skew product on $M \times \mathbb{R}$ is *reducible* (see § 2.1.3 for the definition), $M \times \mathbb{R}$ is foliated into invariant sets for $\Phi^f_{\mathbb{R}}$, on which the dynamics is conjugated to $\psi_{\mathbb{R}}$ on M.

Taking a suitably chosen Poincaré section (see § 2.3.3 for details), the ergodicity of $\Phi^f_{\mathbb{R}}$ is equivalent to the ergodicity of a skew product automorphism T_φ of the strip $I \times \mathbb{R}$, where $I = [0, 1)$, of the form

$$T_\varphi(x, y) = (T(x), y + \varphi(x)), \qquad x \in I, \quad y \in \mathbb{R}, \tag{1.4}$$

where $T : I \to I$ is a *rotation* (i.e. the map $T(x) = x + \alpha \mod 1$) when M is a torus ($g = 1$), or more in general, for any $g \geq 1$, an *interval exchange transformation* (see § 2.2.1), while $\varphi : I \to \overline{\mathbb{R}}$ is a function with *singularities* (i.e. points where the function blows up) which are are *logarithmic* (see § 2.3.1 for the precise definition) whenever $\psi_{\mathbb{R}}$ has only non-degenerate saddles (while *polynomial* in presence of a degenerate saddle).

Remark 1.1. Notice also that if f is compactly supported in $M\backslash\mathrm{Fix}(\psi_{\mathbb{R}})$ (or, more generally, it vanishes on $\mathrm{Fix}(\psi_{\mathbb{R}})$, see § 4.2.2, in particular Proposition 4.1, then the function φ in (1.4) is piecewise absolutely continuous (or even piecewise smooth), in particular does not have logarithmic singularities. Thus, the singularities are a combined effect of the nature of the locally Hamiltonian parametrization, *together with* the assumption that (the jet of) f does not vanish identically zero at $\mathrm{Fix}(\psi_{\mathbb{R}})$.

We stress that the problem of ergodicity of skew product extensions over IETs is currently actively researched, but still widely open. See for example [29, 10, 21, 17, 28, 53, 54, 7] for some results in particular settings.

In the *genus one* case, the existence of ergodic skew products was first discovered by Krygin, in [42], in the case where the flow $\psi_{\mathbb{R}}$ has no singularities. Ergodicity of extensions of typical *Arnold flows*[8] (or, correspondingly, of skew products of the form (1.4) where T is a rotation and φ has one *asymmetric* logarithmic singularity, see § 2.3.1 for definitions), was proved by Fayad and Lemańczyk in [14], where they proved ergodicity for a full measure set of rotation numbers. This case is particularly delicate since the underlying Arnold flows are mixing; in a related easier case (namely the case when T is a rotation but φ in (1.4) has one *symmetric* logarithmic singularity, see § 2.3.1), ergodicity was proved previously by Lemańczyk and the first author, see [19].

Very little is understood in the case of infinite skew product extensions (i.e. extensions by a non-compact fiber, for which the natural invariant measure is infinite) of locally Hamiltonian flows in higher genus $g \geq 2$, even in the case when $f : M \to \mathbb{R}$ has compact support in $M\backslash\mathrm{Fix}(\psi_{\mathbb{R}})$ and the cocycle φ is piecewise-smooth (see Remark 1.1) or even piecewise-constant. Some specific results for piecewise constant or piecewise absolutely continuous cocycles over IETs with $d > 2$ were proved for example in [10, 21, 22, 44].

We considered the case of a locally Hamiltonian flow $\psi_{\mathbb{R}}$ with non-degenerate saddles and a general observable $f : M \to \mathbb{R}$ and, correspondingly, of a cocycle φ with *logarithmic* (symmetric) *singularities* in our previous joint work [20], where we showed the *existence* of ergodic extensions in any genus, but for a very restrictive class of locally Hamiltonian flows. More precisely, in [20] we could treat only the special (measure

[8] Recall that an Arnold flow is the restriction to the minimal component of a locally Hamiltonian flow in genus one with one saddle and one center, see Figure 1(a).

zero) class of locally Hamiltonian flows in $\mathscr{U}_{min}$ for which the Poincaré section can be chosen to be a *self-similar* interval exchange transformation[9] and restrict the observable f to belong to an infinite dimensional (but finite codimension g) space. For extensions of flows in this special class, though, we could provide a complete description of the ergodic behavior and prove a dichotomy between ergodicity and reducibility. One of the main results of this paper is to show that this dichotomy actually holds also for a *full measure* set of such minimal locally Hamiltonian flows (see the Main Theorem 1.2 below).

1.4. **Main results.** One of the main results of this paper is that infinite *ergodic* extensions *exist* in any genus $g \geq 1$ for a *full measure* set of (minimal) locally Hamiltonian flows with non-degenerate fixed points (with respect to the Katok fundamental class for each *stratum*, see § 2.1.1). More precisely, we are able to extend the result previously proved in [20] only for a *measure zero class* of self-similar IETs to a *full measure* set of locally Hamiltonian flows, by proving the following *dichotomy* for the dynamics of the extensions:

Theorem 1.2 (**Ergodic or reducible extensions of locally Hamiltonian flows).** *For a* full measure *set of locally Hamiltonian flows $\psi_{\mathbb{R}}$ with non-degenerate saddles in $\mathcal{U}_{min}$, for any $\epsilon > 0$, for any f in a infinite dimensional (finite codimension) subspace $K \subset \mathscr{C}^{2+\epsilon}(M)$, we have the following dichotomy:*

- *If $\sum_{x\in \mathrm{Fix}(\psi_{\mathbb{R}})} |f(x)| \neq 0$ then the extension $\Phi^f_{\mathbb{R}}$ is ergodic;*
- *If $\sum_{x\in \mathrm{Fix}(\psi_{\mathbb{R}})} |f(x)| = 0$ then the extension $\Phi^f_{\mathbb{R}}$ is reducible.*

We will comment later on the full measure set, which is explicitly described by a new *Diophantine-type* condition (see § 3.2.2 for the definition) as well as on the infinite dimensional (invariant) subspace K (which will be defined as the kernel of g invariant distributions, see § 7.2).

The proof of this ergodicity result takes as starting point our results on deviations of ergodic averages[10] of f, which is of independent interest and we now state. As it is clear from the dichotomy, to produce ergodic extensions one needs to study observables $f : M \to \mathbb{R}$ which *do not vanish* at (at least one) the saddle points[11] in $\mathrm{Fix}(\psi_{\mathbb{R}})$.

For ergodic integrals of (typical) minimal locally Hamiltonian flows in $\mathcal{U}_{min}$ (see Theorem 1.3), as well as for minimal components of (typical) locally Hamiltonian flows in $\mathcal{U}_{\neg min}$ (see Theorem 1.4), we give asymptotic descriptions of the deviation spectrum, as follows.

Theorem 1.3 (**Asymptotic power spectrum of ergodic integrals (minimal case)).** *For a full measure set of locally Hamiltonian flows on M in $\mathscr{U}_{min}$ with non-degenerate saddles, there exist a* power spectrum *$0 < \nu_g < \cdots < \nu_2 < \nu_1 := 1$, where g is the genus of the surface M and, for any $\epsilon > 0$,* invariant distributions *$D_i : C^{2+\epsilon}(M) \to \mathbb{R}$, $i = 1, \ldots, g$, such that, for every $f \in C^{2+\epsilon}(M)$, we have the* asymptotic expansion*:*

$$\int_0^T f(\psi_t(x))\, dt = \sum_{i=1}^{g} D_i(f) u_i(T, x) + \sum_{\sigma\in \mathrm{Fix}(\psi_{\mathbb{R}})} f(\sigma) u_\sigma(T, x) + err_b(f, T, x), \tag{1.5}$$

where, for $1 \leq i \leq g$, u_i are smooth cocycles $u_i : \mathbb{R} \times M \to \mathbb{R}$ over the flow $\psi_{\mathbb{R}}$ such that

$$\limsup_{T\to+\infty} \frac{\log \|u_i(T, \cdot)\|_{L^\infty(M)}}{\log T} = \nu_i, \tag{1.6}$$

while, for $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}})$, u_σ are smooth cocycles $u_\sigma : \mathbb{R} \times M \to \mathbb{R}$ over $\psi_{\mathbb{R}}$ which grow sub-polynomially pointwise and in L^p norm for every $p \geq 1$, i.e. such that

$$\limsup_{T\to+\infty} \frac{\log |u_\sigma(T, x)|}{\log T} = 0 \text{ for a.e. } x \in M, \quad \text{and} \quad \limsup_{T\to+\infty} \frac{\log \|u_\sigma(T, \cdot)\|_{L^p(M)}}{\log T} = 0, \text{for all } p \geq 1, \tag{1.7}$$

and err_b is a uniformely bounded error *term, i.e.*

$$\sup_{t\in\mathbb{R}} \|err_b(f, t, \cdot)\|_{L^\infty} < +\infty. \tag{1.8}$$

Furthermore, for every $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}})$ and for μ-almost every $x \in M$, the values of the cocyle $t \mapsto u_\sigma(t, x)$ are equidistributed *on $\mathbb{R}$, i.e. for any pair of intervals $J_1, J_2 \subset \mathbb{R}$ we have*

$$\lim_{T\to+\infty} \frac{Leb\{t \in [0, T] : u_\sigma(t, x) \in J_1\}}{Leb\{t \in [0, T] : u_\sigma(t, x) \in J_2\}} = \frac{|J_1|}{|J_2|}. \tag{1.9}$$

[9]These IETs are also known as *periodic-type* IETs in the literature, see for example [60]. In [20] we further assume that the periodic-type IET is of *hyperbolic type*, see [20] for details. Explicit examples of locally Hamiltonian flow of hyperbolic periodic type were constructed in [10].

[10]In particular, to prove ergodicity we need to show a form of *tightness* of Birkhoff sums, which, combined with enough *oscillations* thanks to the presence of logarithmic singularities, allows to apply classical *essential values* (see [59]).

[11]Since we are here assuming that $\psi_{\mathbb{R}} \in \mathcal{U}_{min}$ has only non-degenerate fixed points, $\mathrm{Fix}(\psi_{\mathbb{R}})$ consists of simple saddles only, see § 2.1.1.

Finally, if we set

$$err(f,t,x) := \sum_{\sigma\in \mathrm{Fix}(\psi_{\mathbb{R}})} f(\sigma)u_\sigma(t,x) + err_b(f,t,x), \tag{1.10}$$

as soon as f does not vanish identically on $\mathrm{Fix}(\psi_{\mathbb{R}})$, for μ-almost every $x \in M$ also the values of the cocyle $t \mapsto err(f,t,x)$ are equidistributed *on $\mathbb{R}$.*

Main Theorem 1.3 completes in particular the proof of the Kontsevich-Zorich conjecture, in its original formulation for smooth functions over locally Hamiltonian flows with non-degenerate saddles (as formulated in [40], see the above § 1.2). The result should be seen as a generalization (for smooth[12] functions) of both the results by Forni [23] (since it proves the existence of a power deviation spectrum) and Bufetov [6] (since we show the existence of asymptotic cocycles). While the observables in both Forni's [23] and Bufetov's [6] works vanish on $\mathrm{Fix}(\psi_{\mathbb{R}})$, we allow the observables to be non-zero at singularities in $\mathrm{Fix}(\psi_{\mathbb{R}})$. This leads to the presence in the asymptotic expansion of k new cocycles, where k is the cardinality of $\mathrm{Fix}(\psi_{\mathbb{R}})$, one for each saddle $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}})$. We will call these u_σ *singular cocycles*, since they describe the fluctuations of the ergodic averages due to the presence of singularities. While these cocycles u_σ have sub-polynomial deviations, as shown by (1.7), they are *not* uniformly bounded.

Comparison to Forni's and Bufetov's works. To further compare the result with Forni's [23] and Bufetov's [6] works, let us consider the global error term $err(f,t,\cdot)$ defined as in (1.10) combining the bounded error $err_b(f,t,\cdot)$ together with the cocycles u_σ, $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}})$. Then one can see that $err(f,t,\cdot)$ has always *sub-polynomial* pointwise growth (in view of (1.7) combined with (1.8)), but we have a *dichotomy*: on one hand, if f does vanish identically on $\mathrm{Fix}(\psi_{\mathbb{R}})$, $err(f,t,\cdot)$ coincides with $err_b(f,t,\cdot)$ and is uniformly bounded. In this case, the g cocycles u_i, which lead the power growth, can be shown *a posteriori* to coincide with the *Bufetov functionals* in [6] up to a bounded error. On the other hand, as soon as f does *not* vanish identically on $\mathrm{Fix}(\psi_{\mathbb{R}})$, $err(f,t,x)$ *cannot* be controlled *uniformly*: for μ-almost every x, the function $t \mapsto err(f,t,x)$ is unbounded, in view of the equidistribution of $err(f,t,\cdot)$ in this case (see the final part of Theorem 1.3, which follows directly from the ergodicity of the extensions proved in the Main Theorem 1.2, more precisely from an application of the *ratio* ergodic theorem in infinite ergodic theory).

This novel phenomenon is an effect of the presence of infinite *tails*, due to the assumption that f is non-zero at (some) singularities and the slowing down of trajectories near Hamiltonian saddles. We are nevertheless able to control the error term $err(f,t,\cdot)$ *pointwise* almost everywhere (in view of (1.12)) and *in average*, in any L^p norm with $p \geq 1$, in view of (1.13).

Minimal components in the non-minimal setting. Another novelty of our work is that, while Forni and Bufetov in [23, 6] study only minimal flows, we prove the existence of an asymptotic expansion also for ergodic integrals of non-mimimal flows in $\mathcal{U}_{\neg min}$. More precisely, we prove the following result for a minimal component $M_0 \subset M$ of a *typical* flow on $\mathcal{U}_{\neg min}$.

Theorem 1.4 (**Asymptotic power spectrum for non minimal components).** *For a full measure set of locally Hamiltonian flows on M in $\mathscr{U}_{\neg min}$ with non-degenerate saddles, for any minimal component $M_0 \subset M$ of $\psi_{\mathbb{R}}$, if g_0 denotes the genus of M_0, there exist a* power spectrum $0 < \nu_{g_0} < \cdots < \nu_2 < \nu_1 := 1$ *and, for any $\epsilon > 0$,* g_0 invariant distributions $D_i : C^{2+\epsilon}(M_0) \to \mathbb{R}$, $i = 1,\dots,g_0$, *and* g_0 *smooth cocycles* $u_i : \mathbb{R}\times M_0 \to \mathbb{R}$, *for $i = 1,\dots,g_0$, each of which satisfies* (1.6)*, such that for every $f \in C^{2+\epsilon}(M_0)$ we have an asymptotic expansion*

$$\int_0^T f(\psi_t(x))\,dt = \sum_{i=1}^{g_0} D_i(f)u_i(T,x) + err(f,T,x),$$

where, if f vanishes on $\mathrm{Fix}(\psi_{\mathbb{R}}) \cap M_0$, the error term $err(f,T,\cdot)$ satisfies

$$\limsup_{T\to+\infty} \frac{\log \|err(f,T,\,\cdot\,)\|_{L^\infty(M_0)}}{\log T} \leq 0, \tag{1.11}$$

while if f is not identically zero on $\mathrm{Fix}(\psi_{\mathbb{R}}) \cap M_0$ then

$$\limsup_{T\to+\infty} \frac{\log |err(f,T,x)|}{\log T} = 0 \text{ for } \mu\text{-almost every } x \in M_0, \tag{1.12}$$

and furthermore

$$\limsup_{T\to+\infty} \frac{\log \|err(f,T,\,\cdot\,)\|_{L^p(M_0)}}{\log T} = 0 \text{ for every } p \geq 1. \tag{1.13}$$

[12]The class of functions considered by Forni [23, 25] and Bufetov [6] is more general: smoothness is not required, but only a Sobolev condition in [23] (see also [25] for a more general result on the cohomological equation) and a *weak Lipschitz property* in Bufetov's work, see [6] for details.

Notice that in this case, when restricting to a minimal component of $\psi_{\mathbb{R}} \in \mathcal{U}_{\neg min}$, we only claim that $err(f,T,\cdot)$ grows sub-polynomially (which is the same type of estimate proved by Bufetov for the error term in the symmetric case). This result is in particular an extension of Bufetov's work [6] to the restriction to a minimal component in the non minimal case $\psi_{\mathbb{R}} \in \mathcal{U}_{\neg min}$.

Thus, Theorems 1.3 and 1.4 complete the study of deviations of ergodic averages of smooth functions over locally Hamiltonian flows with *non-degenerate* saddles. The study of locally Hamiltonian flows with *degenerate-saddles* leads to other new phenomena and additional polynomial terms in the asymptotic expansion and is treated in an upcoming paper by M. Kim and the first author [18].

On the proof and the Diophantine-like conditions. The proof of the asymptotic expansion in Theorem 1.3, which will be proved at the same time than Theorem 1.4, follows a completely different approach to both Forni's [23] and Bufetov's [6] works and is inspired by Marmi-Moussa-Yoccoz work [44] on solving the cohomological equation for (Roth-type) interval exchange transformations (and the follow up work [48] by Marmi and Yoccoz). We comment in detail on this strategy below in § 1.5.

An advantage of this different approach is that it allows to give a *description* of the full measure set of locally Hamiltonian flows for which the result holds in terms of a *Diophantine-type* condition. Furthermore, it also provides a different construction of the cocycles which describe the asymptotic behaviour of ergodic integrals in terms of the correction operators.

The full measure *Diophantine-like* conditions (which are different for Theorem 1.3 and Theorem 1.4 respectively) are expressed more precisely on the interval exchange transformations which arise as Poincaré sections of the flows. We introduce (in § 3.2) two such conditions, both of which we show to be of full measure. The first, that we call *Uniform Diophantine Condition* (or UDC), is used to prove the existence of the asymptotic expansion in both Theorem 1.3 and Theorem 1.4 up to a subpolynomial error. In the case of minimal flows in $\mathcal{U}_{min}$, to improve the estimates on the error and show in particular that the error is equidistributed (see the second part of Theorem 1.3), we need to assume a more restrictive condition, namely the *Symmetric Uniform Diophantine Condition* (or SUDC). For this result indeed we also need to crucially exploit the cancellations proved by the second author in [63] to prove typical absence of mixing and these require further assumptions on the IET to hold.

Both Diophantine-like conditions expressed in terms of the matrices of the Rauzy-Veech cocycle, which often plays the role of *multi-dimensional continued fraction* in the study of IETs. These conditions, similarly to the Roth-type condition for IETs introduced by Marmi-Moussa and Yoccoz in [44] (and its variations, see for example [44, 46, 48, 47]), impose constraints both on the growth of the matrices of (an acceleration of) the cocycle, as well as requests on the hyperbolic behaviour of the matrix product, in the form of Oseledets genericity requests. In addition, we require *effective* Oseledets control, which in turns allow to control certain *Diophantine series* (see § 3.3). We point out that similar conditions also appear in the recent work [27] on rigidity of generalized interval exchanges.

1.5. **Correction of cocycles with logarithmic singularities.**

We comment now on the methods and the proofs. First of all we work with Poincaré maps, both to study the flow $\psi_{\mathbb{R}}$ and its extensions $\Phi^f_{\mathbb{R}}$; it is well known that Poincaré maps of area-preserving flows, in suitably chosen coordinates, are *interval exchange transformations* (for short IETs), namely, piecewise-isometries of the interval $I = [0, 1)$ (the definition is recalled in § 2.2.1). Moreover, any minimal locally Hamiltonian flow admits a representation as *special flow* over the IET $T : I \to I$ which arise as Poincaré map (see § 2.2.2 for definitions). The *roof function* $r : I \to \overline{\mathbb{R}^+}$ which arise from this representation has *singularities* at the discontinuities of T, which, in case of simple (non-degenerate) saddles, are of *logarithmic type* (formally defined in § 2.3.1), i.e. as $x \to x_i^{\pm}$ approaches a discontinuity $x_i \in I$ of T from the right or left, $r(x)$ blows up as $C_i^{\pm}|\log(x - x_i)|$, where the constants $C_i^{\pm}$ are *positive* and are globally *symmetric*, namely $\sum C_i^+ = \sum C_i^-$, for typical flows in $\mathcal{U}_{min}$, while *asymmetric* for minimal components of typical flows in $\mathcal{U}_{\neg min}$.

Fix now an observable $f : M \to \mathbb{R}$ which is non-zero on $\mathrm{Fix}(\psi_{\mathbb{R}})$. To study ergodic integrals, we build the extension $\Phi^f_{\mathbb{R}}$ on $M \times \mathbb{R}$ (given by (1.3)). Choosing a Poincaré section for the extension which projects on I, namely of the form $I \times \mathbb{R}$, the Poincaré first return map of $\Phi^f_{\mathbb{R}}$ (in suitable coordinates) turns out to be a *skew product* over the IET T of the form (1.4), in which the *cocycle* φ has *logarithmic singularities* (where the constants $C_i^{\pm}$ here can be positive or negative, or zero if the function is zero on $\mathrm{Fix}(\psi_{\mathbb{R}})$, in which case there are no singularities, see Remark 1.1). We have now reduced the study of ergodic integrals and ergodicity of extensions to the study of Birkhoff sums of cocycles with logarithmic singularities over IETs and ergodicity of skew products over IETs with logarithmic singularities.

Under the new Diophantine-type conditions that we introduce in § 3.2, for every function with logarithmic singularities, we prove the existence of a *correction* operator, namely an operator which, removing the projection on a finite dimensional space (which corresponds morally to the projection on the unstable space of renormalization), allows to get a better control of the behaviour of Birkhoff sums of functions with logarithmic singularities (see Theorem 6.1 for the precise statement). The result provides an extension of the main result in the work of Marmi-Moussa-Yoccoz [44]. In the latter, in order to solve the cohomological equation for IETs of Roth-type, correction operators are constructed for (piecewise) *absolutely continuous* cocycles.

While the main steps of our correction procedure are inspired by the construction introduced in [44] (and later developed in [48]), there are considerable differences and difficulties. Notably, while the authors of [44] were interested in controlling the growth of the sequence of Birkhoff sums $(S(k)\varphi)_{k\in\mathbb{N}}$ of a piecewise absolutely continuous functions φ using the *uniform norm* (in order to keep them bounded after correction and be able to apply Gottschalk-Hedlund theorem, see [44] for details), for functions with *logarithmic singularities*, the uniform norm *cannot* be used (since functions with logarithmic singularities are always unbounded). The key idea to treat cocycles with logarithmic singularities in this paper is to exploit instead the L^1-norm and to build correction operators which allow to bound or control the L^1-norm of the sequence $(S(k)\varphi)_{k\in\mathbb{N}}$. The use of the L^1-norm has already appeared in our previous work [20], where we had considered the correction problem[13] for the (measure zero set of) IETs of hyperbolic periodic type. It turns out that to extend the result to almost every IET requires once again changes in the basic step of construction, as well as the introduction of the above mentioned delicate Diophantine-type condition on the IET. We refer the interested reader to § 6 (and in particular the outline of the strategy to build the correction operators given in § 6.1.2) for further details on the differences and the steps in the construction of the correction operators.

The construction of the asymptotic cocycles $u_i : \mathbb{R} \times M \to \mathbb{R}$ which lead to understanding the behaviour of ergodic integrals (see the statement of Main Theorem 1.3) is strictly connected to the finite dimensional space of *corrections*. Indeed, corrections can be realized by subtracting *piecewise constant cocycles*, which, through the correspondence between extensions and skew-products, allow to define the asymptotic cocycles u_i.

In the case of minimal locally Hamiltonian flows in $\mathcal{U}_{min}$ (which give rise to *symmetric* logarithmic singularities), we also exploit the delicate *cancellations* among contributions of singularities which were proved by the second author in [63] and, introducing the SUDC Diophantine-type condition, we are able to prove that, after corrections, a subsequence of Birkhoff sums $(S(k)\varphi)_{k\in\mathbb{N}}$ is *tight*. Tightness, combined with partial rigidity of the IET in the base (a result which dates back to Katok [33]) and the *presence* of logarithmic singularities (which comes from the assumption that f is non identically zero on $\mathrm{Fix}(\psi_{\mathbb{R}})$), allows to apply a quite standard ergodicity criterium based on the existence of *essential values* (see Proposition 8.4 for the precise incarnation of the criterium which we use in this paper). This allows to prove ergodicity of the corresponding extensions.

Structure of the paper. In § 2 we recall basic definitions and background material on locally Hamiltonian flows and their extensions. We also summarize their typical ergodic properties and explain the reductions to special flows and skew products over IETs. In § 3, after recalling the required definitions and properties of the Rauzy-Veech induction procedure and the associated cocycle, we define the two Diophantine conditions (the UDC and the SUDC conditions) and prove that they have full measure.

In §§ 4, 5 and 6 we study cocycles with logarithmic singularities over IETs. After giving definitions and proving elementary properties in § 4, we proceed in § 5 at investigating the renormalization process induced on such cocycles by performing Rauzy-Veech induction. The correction operators are constructed in § 6 (where the above mentioned Theorem 6.1 about existence and properties of the correction operators is proved).

The asymptotic deviation spectrum (see the first part of Main Theorem 1.3) is proved in § 7.2, where the asymptotic of ergodic integrals is recovered from the cocycles associated to the correction operators. In § 8 we state the ergodicity criterium that we then apply to prove ergodicity of extensions. After discussing also the reducibility case, we then prove Main Theorem 1.2, as well as the second part of Main Theorem 1.3. Some technical but standard proofs in this part are relegated to the Appendix (in particular the proofs of the ergodicity criterium and of a cohomological reduction result which is needed for the reducibility part).

2. Definitions, background material and reductions

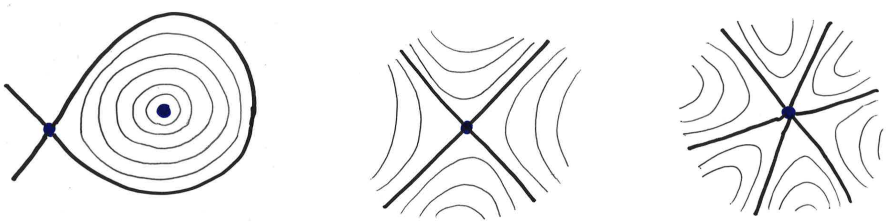
In this section we recall some basic definitions and background material concerning locally Hamiltonian flows (§ 2.1) and their extensions (§ 2.1.3), including a brief summary in § 2.1.4 of our current knowledge of their typical chaotic properties. We also the definition of special flows (see § 2.2.2) and skew-products (in § 2.2.3) over interval exchange transformations (defined in § 2.2.1). We finally recall in § 2.3 the representation of locally Hamiltonian flows to special flows (see § 2.3.2) with logarithmic singularities (defined in § 2.3.1) and the reduction of the study of their extensions to skew products over IETs, see § 2.3.3.

[13] In [20], for IETs of hyperbolic periodic type, we build correction operators for cocycles with *symmetric* logarithmic singularities and we then exploit the result to build ergodic extensions, but we do not work out the full deviation spectrum and asymptotic cocycles formalism.

2.1. **Locally Hamiltonian flows.** Let (M, ω) be a surface with a fixed smooth area form ω. A *smooth area preserving flow* $\psi_{\mathbb{R}} = (\psi_t)_{t \in \mathbb{R}}$ on M is a smooth flow on M which preserves the measure μ associated to ω. These flows are also called *locally Hamiltonian flows* or *multi-valued Hamiltonian flows* in the literature, in view of their interpretation as flows locally given by Hamiltonian equations, see the introduction.

It turns out that such smooth area preserving flows on M are in one-to-one correspondence with smooth closed real-valued differential 1-forms as follows. Given a smooth, closed, real-valued differential 1-form η, let X be the vector field determined by $\eta = i_X\omega$ where i_X denotes the contraction operator, i.e. $i_X\omega = \omega(\eta, \cdot)$ and consider the flow $\psi_{\mathbb{R}}$ on M given by X. Since η is closed, the transformations ψ_t, $t \in \mathbb{R}$, are area-preserving. Conversely, every smooth area-preserving flow can be obtained in this way.

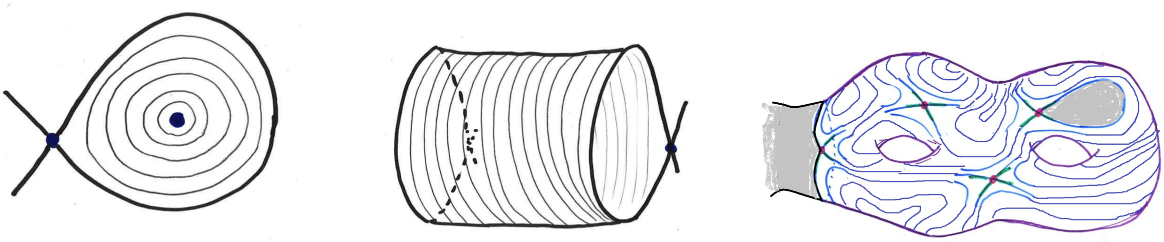
Let $\mathrm{Fix}(\psi_{\mathbb{R}})$ denote the set of *fixed points* (also called *singularities*) of the flow $\psi_{\mathbb{R}}$. We will always require that $\mathrm{Fix}(\psi_{\mathbb{R}})$ is a *finite set*, so in particular singularities are *isolated*. Remark that when $g \geq 2$, $\mathrm{Fix}(\psi_{\mathbb{R}})$ is always not empty, thus singularities are isolated. Since $\psi_{\mathbb{R}}$ is area-preserving, *singularities* in $\mathrm{Fix}(\psi_{\mathbb{R}})$, as shown in Figure 2, can be either centers (Fig. 2(a)), simple saddles (Fig. 2(b)) or multi-saddles (i.e. saddles with $2k$ pronges, $k \geq 2$, see Fig. 2(c) for $k = 3$). For $g = 1$, i.e. on a torus, if there is a singularity then there has to be another one and we get an Arnold flow as in Figure 1(a).



(a) center (b) simple saddle (c) multisaddle

FIGURE 2. Type of singularities of a locally Hamiltonian flow.

We call *saddle connection* a flow trajectory from a saddle to a saddle and a *saddle loop* a saddle connection from a saddle to the same saddle (see Fig. 3). A *periodic component* is either a (maximal) punctured disk or a (maximal) cylinder filled with closed (i.e. periodic) trajectories (see Fig. 3(a) and Fig. 3(b) respectively). A *minimal component* is a subsurface $M' \subset M$, possibly with boundary, such that any trajectory different than a fixed point is *dense* in M'. Periodic and minimal components are bounded by union of saddle connections.



(a) periodic island (b) periodic cylinder (c) $g = 2$ minimal component

FIGURE 3. Periodic and minimal components.

2.1.1. *Open sets, genericity and minimality.* Let us denote by $\mathcal{F}$ the set of smooth closed 1-forms on M (i.e. locally Hamiltonian flows) with isolated zeros. One can define a *topology* on $\mathcal{F}$ by considering perturbations of closed smooth 1-forms by (small) closed smooth 1-forms[14]. We say that a condition is *generic* (in the sense of Baire) if it holds for flows described by an open and dense set of forms with respect to this topology.

Let $\mathcal{A} \subset \mathcal{F}$ be the subset of Morse 1-forms (adopting the notation introduced by Ravotti [56]), namely forms which are locally the differential of a *Morse function* (i.e. a function that has *non-degenerate* zeros, so that the Hessian at every fixed point is non-degenerate). The set $\mathcal{A}$ of Morse 1-forms is then *generic*. Locally Hamiltonian flows corresponding to forms in $\mathcal{A}$ have only *non-degenerate fixed points*, i.e. *centers* and *simple saddles* (see Figures 2(a) and 2(b)), as opposed to degenerate *multi-saddles* (as in Fig. 2(c)). We

[14]Let η, η' be two smooth closed 1-forms. We say that η' is an ϵ-perturbation of η if for any $x \in M$ there exists coordinates on a simply connected neighbourhood U of x, such that $\eta|_U = dH$ and $(\eta' - \eta)|_U = dh$ where $\|h\|_{C^\infty} < \epsilon \|H\|_{C^\infty}$.

denote by $\mathcal{A}_{s,c}$ the set of 1-forms in $\mathcal{A}$ with s saddle points and c centers. By the Poincare-Hopf Theorem, $c - s = 2 - 2g$. Furthermore, each $\mathcal{A}_{s,l}$ is open and their union $\mathcal{A}$ is dense in $\mathcal{F}$ (see e.g. Lemma 2.3 in [56]).

For every 1-form in $\mathcal{A}$, the surface M splits into periodic components and (up to g) minimal components (as proved independently by Maier [43], Levitt [41] and Zorich [72]). Notice that if there is a unique minimal component (which is equal to the whole surface M), then $c = 0$ (since if there is a center is associated to a periodic component) and $s = 2g - 2$.

Moreover, one can show that if the flow $\psi_{\mathbb{R}}$ given by a closed 1-form η has a *saddle loop homologous to zero* (i.e. the saddle loop is a *separating* curve on the surface), then the saddle loop is persistent under small perturbations (see § 2.1 in [72] or Lemma 2.4 in [56]). In particular, the set of locally Hamiltonian flows which have at least one saddle loop is an open set, which consists of *non-minimal* flows. The set $\mathscr{U}_{\neg min}$ mentioned in the introduction is an open and dense set of this open set (where the open condition guarantees *asymmetry* in the special flow representation recalled in § 2.3.2, we refer to [56] for the precise definition, see Notation 3.3 in § 3.1 of [56]). The set $\mathscr{U}_{min}$ is given by the interior (which one can show to be non-empty) of the complement of $\mathscr{U}_{\neg min}$, i.e. the set of locally Hamiltonian flows without saddle loops homologous to zero[15].

2.1.2. *Measure class and typicality.* Let us fix an open set $\mathcal{A}_{s,c}$ of closed 1-forms with c centers and s (simple) saddles. A *measure-theoretical notion of typical* on $\mathcal{A}_{s,c}$ can be defined on each $\mathcal{A}_{s,c}$ as follows, by using the *Katok fundamental class* (introduced by Katok in [32], see also [50]), i.e. the cohomology class of the 1-form η which defines the flow. Let $\gamma_1, \ldots, \gamma_n$ be a base of the relative homology $H_1(M, \mathrm{Fix}(\psi_{\mathbb{R}}), \mathbb{R})$, where $n = 2g + s + c - 1$, and consider the period map

$$\Theta(\eta) = \Big(\int_{\gamma_1} \eta, \ldots, \int_{\gamma_n} \eta \Big) \in \mathbb{R}^n.$$

The map Θ is well defined in a neighbourhood of η in $\mathcal{A}_{s,c}$ and one can show that it is a complete isotopy invariant (see [32], or also Prop. 2.7 in [56]).

The pull-back $Per_* Leb$ of the Lebesgue measure class (i.e. class of sets with zero measure) by the period map gives the desired measure class on closed 1-forms in $\mathcal{A}_{s,c}$. When we use the expression *typical* below (or typical in $\mathcal{U}_{min}$ or $\mathcal{U}_{\neg min}$) we mean full measure in each $\mathcal{A}_{s,c}$ with respect to this measure class on each $\mathcal{A}_{s,c}$ (or on each open subset of $\mathcal{A}_{s,c}$ contained in the union $\mathcal{U}_{min}$ or $\mathcal{U}_{\neg min}$).

2.1.3. *Ergodicity and reducibility of extensions.* Let $\Phi^f_{\mathbb{R}} := (\Phi^f_t)_{t\in\mathbb{R}}$ on $M \times \mathbb{R}$ denotes the extension of an ergodic flow $\psi_{\mathbb{R}}$ on M by $f : M \to \mathbb{R}$ given by the formula (1.3). Recall that, if $\psi_{\mathbb{R}}$ preserves a measure μ, $\Phi^f_{\mathbb{R}}$ preserves the (infinite) measure $\mu \times Leb$. The flow $\Phi^f_{\mathbb{R}}$ is *recurrent* if $\mu \times Leb$-almost every point is recurrent. A result by Atkinson [3] (which holds for 1-dimensional extensions of ergodic flows) shows that $\Phi^f_{\mathbb{R}}$ is recurrent if and only if $\int_M f d\mu = 0$.

We recall that $\Phi^f_{\mathbb{R}}$ is *ergodic* with respect to the (infinite) measure $\mu \times Leb$ if for any measurable set A which is invariant, i.e. such that $\mu \times Leb(A) = \mu \times Leb(\Phi^f_t A)$ for all $t \in \mathbb{R}$, either $\mu \times Leb(A) = 0$ or $\mu \times Leb(A^c) = 0$, where A^c denotes the complement.

Remark that if $f = 0$, the phase space $M \times \mathbb{R}$ for the corresponding trivial extension given by $\Phi^f_t(x, y) = (\psi_t(x), y)$ is foliated in invariant sets of the form $M \times \{y\}$, $y \in \mathbb{R}$. In this sense, the dynamics is reduced to the dynamics of the surface flow $\psi_{\mathbb{R}}$. We say that $\Phi^f_{\mathbb{R}}$ is (topologically) reducible if it is isomorphic to $\Phi^0_{\mathbb{R}}$ and the isomorphism $G : M \times \mathbb{R} \to M \times \mathbb{R}$ is of the form $G(x, y) = (x, y + g(x))$, where $g : M \to \mathbb{R}$ is continuous. So the reducibility of $\Phi^f_{\mathbb{R}}$ is equivalent to asking that

$$\int_0^t f(\psi_s x)\, ds = g(x) - g(\psi_t x)$$

for every regular point $x \in M$ and any $t \in \mathbb{R}$. In this case, the phase space is again foliated into invariant sets for $\Phi^f_{\mathbb{R}}$ of the form $\{(x, y + g(x)),\ x \in M\},\ \ y \in \mathbb{R}$. On each leaf the action of $\Phi^f_{\mathbb{R}}$ is conjugated to $\psi_{\mathbb{R}}$ on M.

2.1.4. *Typical chaotic properties of locally Hamiltonian flows.* Let us briefly summarize the key chaotic properties of locally Hamiltonian flows and some of the recent works on this topic. We already recalled in the introduction, in view of the relation between locally Hamiltonian flows and translation flows (see also Remark 2.3), the seminal works by Keane [36] and Masur [49] and Veech [64] show that a full measure set of locally Hamiltonian flows in $\mathcal{U}_{min}$ are minimal and ergodic and that almost every flow in $\mathcal{U}_{\neg min}$, the restriction to each minimal component is ergodic (and in both cases the underlying foliation in uniquely ergodic).

[15]Note that saddle loops *non-homologous* to zero (as well as saddle connections) disappear after arbitrarily small perturbations; therefore neither the set of 1-forms with saddle loops (or more generally saddle connections) non-homologous to zero, nor its complement are open (see [56] for details).

Mixing depends crucially on the type of singularities of the flow. For a (non-generic) locally Hamiltonian flow with at least one *degenerate saddle* (see e.g. Figure 2(c)), mixing was proved in the 1970*s* (by Kochergin in [39]). When $\eta \in \mathcal{A}$ and all saddles are *simple*, one has the following dichotomy: in $\mathscr{U}_{min}$, the typical locally Hamiltonian flow is *weakly mixing, but it is* not *mixing* in view of work [62, 63] by the second author (see also [37, 38] and [58] for previous special cases of this result). There exist nevertheless exceptional mixing flows, see the work by [8], which produces sporadic examples in $g = 5$. If $\eta \in \mathcal{U}_{\neg min}$, the restriction of the typical locally Hamiltonian flow $\psi_{\mathbb{R}}$ on each of its minimal components is mixing (as proved by Ravotti [56] extending previous work by the second author [61]). Ravotti also shows in [56] *subpolynomial* bounds for the speed of mixing.

Further recent work (see [30]) also shows that locally Hamiltonian flows in $\mathcal{U}_{\neg min}$ display a *quantitative shearing* property inspired by the *Ratner property* which plays a crucial role in the theory of unipotent flows (or more precisely a variation introduced in [13] to deal with the presence of singularities). From this property, one can deduce that the restriction of a typical locally Hamiltonian flow $\psi_{\mathbb{R}}$ in $\mathscr{U}_{\neg min}$ on its minimal components is not only mixing, but *mixing of all orders*, see [30]. Arnold flows in genus one were also recently shown (by A. Kanigowski and M. Lemańczyk and the second author, see [31]) to typically have *disjointess*[16] *of rescalings*, a property which in particular implies Sarnak Möbius orthogonality conjecture [57] to hold (see [31] for details and [15] for a nice survey on the conjecture and progress toward it).

The spectral theory of locally Hamiltonian flows is still largely not understood. Examples[17] of locally Hamiltonian flows on surfaces of any genus ≥ 1 with singular continuous spectrum were build by M. Lemańczyk and the first author (see [19, Theorem 1]). For some flows in genus one with a *degenerate* singularity (sometimes known as *Kochergin flows*), Forni, Fayad and Kanigowski could recently, prove in [12] that the spectrum is *countably Lebesgue.* The first typical spectral result for surfaces of higher genus, namely $g \geq 2$ was recently proved by Chaika, Kanigowski and the authors, who showed in [9] that a *typical* locally Hamiltonian flow on a *genus two* surface with two isomorphic *simple saddles* has *purely singular* spectrum.

2.2. **IETs, special flows and extension.** Let us now introduce the notation that we will use for interval exchange transformations (§ 2.2.1) and recall the definition of two basic constructions, special flows (§ 2.2.2) and extensions of IETs (§ 2.2.3).

2.2.1. *Interval exchange transformations.* Let $\mathcal{A}$ be a d-element alphabet and let $\pi = (\pi_0, \pi_1)$ be a pair of bijections $\pi_\varepsilon : \mathcal{A} \to \{1, \dots, d\}$ for $\varepsilon = 0, 1$. We adopt the notation from [66]. Denote by $\mathcal{S}^0_{\mathcal{A}}$ the subset of irreducible pairs, i.e. such that $\pi_1 \circ \pi_0^{-1}\{1, \dots, k\} \neq \{1, \dots, k\}$ for $1 \leq k < d$.

For any $\lambda = (\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}_{>0}$ let

$$|\lambda| = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha, \quad I = [0, |\lambda|)$$

and define

$$I_\alpha = [l_\alpha, r_\alpha), \text{ where } l_\alpha = \sum_{\pi_0(\beta)<\pi_0(\alpha)} \lambda_\beta, \quad r_\alpha = \sum_{\pi_0(\beta)\leq\pi_0(\alpha)} \lambda_\beta.$$

Then $|I_\alpha| = \lambda_\alpha$. Denote by Ω_π the matrix $[\Omega_{\alpha\,\beta}]_{\alpha,\beta\in\mathcal{A}}$ given by

$$\Omega_{\alpha\,\beta} = \begin{cases} +1 & \text{if } \pi_1(\alpha) > \pi_1(\beta) \text{ and } \pi_0(\alpha) < \pi_0(\beta), \\ -1 & \text{if } \pi_1(\alpha) < \pi_1(\beta) \text{ and } \pi_0(\alpha) > \pi_0(\beta), \\ 0 & \text{in all other cases.} \end{cases}$$

Given $(\pi, \lambda) \in \mathcal{S}^0_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_{>0}$ let $T_{(\pi,\lambda)} : [0, |\lambda|) \to [0, |\lambda|)$ stand for the *interval exchange transformation* (IET) on d intervals I_α, $\alpha \in \mathcal{A}$, which are rearranged according to the permutation $\pi_1^{-1} \circ \pi_0$, i.e. $T_{(\pi,\lambda)}x = x + w_\alpha$ for $x \in I_\alpha$, where $w = \Omega_\pi\lambda$.

Keane condition. Let $End(T)$ stand for the set of end points of the intervals $I_\alpha : \alpha \in \mathcal{A}$. A pair (π, λ) satisfies the *Keane condition* if $T^m_{(\pi,\lambda)} l_\alpha \neq l_\beta$ for all $m \geq 1$ and for all $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$. Keane [36] showed that an IET with an irreducible permutation that satisfy the Keane condition is *minimal.*

We record here two remarks that will be useful later.

Remark 2.1. Note that for every $\alpha \in \mathcal{A}$ with $\pi_0(\alpha) \neq 1$ there exists $\beta \in \mathcal{A}$ such that $\pi_0(\beta) \neq d$ and $l_\alpha = r_\beta$. It follows that

$$\{l_\alpha : \alpha \in \mathcal{A},\ \pi_0(\alpha) \neq 1\} = \{r_\alpha : \alpha \in \mathcal{A},\ \pi_0(\alpha) \neq d\}.$$

Remark 2.2. Denote by $\widehat{T}_{(\pi,\lambda)} : (0, |I|] \to (0, |I|]$ the exchange of the intervals $\widehat{I_\alpha} := (l_\alpha, r_\alpha]$, $\alpha \in \mathcal{A}$, i.e. $T_{(\pi,\lambda)}x = x + w_\alpha$ for $x \in (l_\alpha, r_\alpha]$. Note that for every $\alpha \in \mathcal{A}$ with $\pi_1(\alpha) \neq 1$ there exists $\beta \in \mathcal{A}$ such that $\pi_1(\beta) \neq d$ and $T_{(\pi,\lambda)} l_\alpha = \widehat{T}_{(\pi,\lambda)} r_\beta$.

[16] The notion of *disjointness* in ergodic theory was introduced in the 1970*s* by H. Furstenberg, see in particular [26].

[17] These examples are known as *Blokhin examples* and are essentially built glueing genus one flows. This allows to study them using (special flows over) rotations. On the other hand, they are highly non typical.

2.2.2. *Special flow definition.* Let $T : I \to I$ be an (ergodic) IET and let $r : I \to \mathbb{R}_{>0} \cup \{+\infty\}$ be an integrable function such that $\underline{r} = \inf_{x\in I} r(x) > 0$. The *special flow* over T under the *roof function* r is the flow $T^r_{\mathbb{R}} := (T^r_t)_{t\in\mathbb{R}}$ acting on

$$I^r := \{(x,s) \in I \times \mathbb{R} : 0 \leq s < r(x)\},$$

so that $T^r_t(x,s) = (x, s+t-r^{(n)}(x))$, where $r^{(n)}(x)$ denote the Birkhoff sums cocycle[18] associated to r and n is the unique integer number with $r^{(n)}(x) \leq s+t < r^{(n+1)}(x)$. It describes the motion of a point in $(x,s) \in I^r \subset I \times \mathbb{R}$ along vertical trajectories, modulo the identification of each point $(x, r(x))$, $x \in I$, with the point $(Tx, 0)$.

2.2.3. *Skew product extensions.* Given an IET $T : I \to I$ and a function $\varphi : I \to \mathbb{R}$ the *extension* of T by φ is the skew-product map $T_\varphi : I \times \mathbb{R} \to I \times \mathbb{R}$ defined as in (1.4) by $T_\varphi(x,y) = (T(x), y + \varphi(x))$. Notice that, for $n \geq 0$, the iterates of T_φ have the form

$$T^n_\varphi(x,y) = (T^n(x), y + \varphi^{(n)}(x)), \qquad \text{where} \quad \varphi^{(n)}(x) := \sum_{k=0}^{n-1} \varphi(T^k(x)).$$

Remark that the Birkhoff sums $\varphi^{(n)}(\cdot)$ are a (additive) *cocycle* over T in view of the *cocycle relation* $\varphi^{(m+n)}(x) = \varphi^{(m)}(T^n x) + \varphi^{(n)}(x)$.

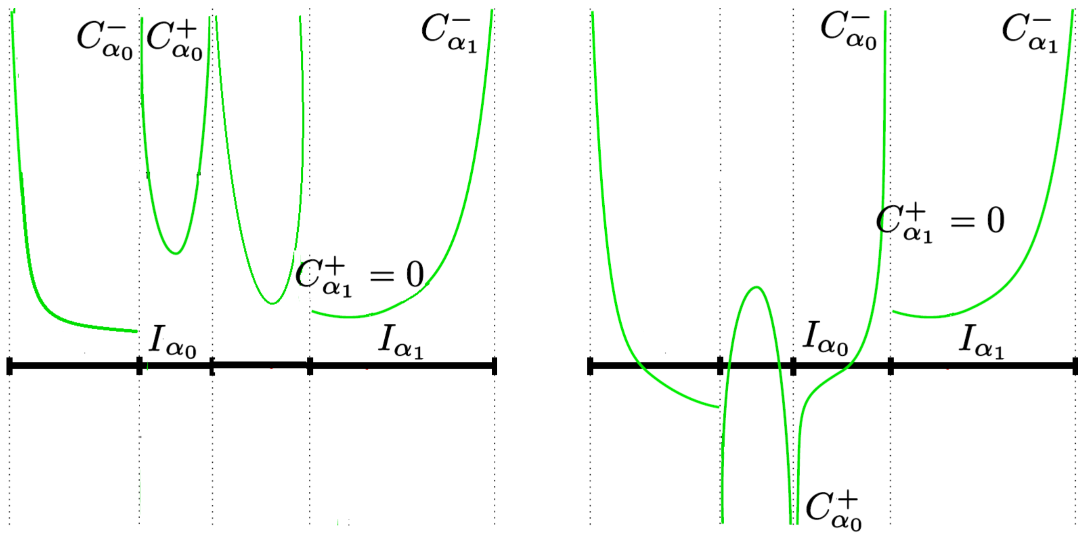
2.3. **Reduction to special flows and skew-product presentations.** We recall two classical results that show that locally Hamiltonian flows and their extensions can be reduced respectively to the study of special flows and skew-product extensions over IETs, with roof functions or, respectively, cocycles, with logarithmic singularities.

2.3.1. *Logarithmic singularities.* We say that a function (or cocycle) $\varphi : I \to \mathbb{R}$ for an IET $T_{(\pi,\lambda)}$ has *logarithmic singularities* if there exist constants $C^+_\alpha, C^-_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{A}$, and a function g_φ absolutely continuous on the interior of each interval I_α, $\alpha \in \mathcal{A}$ (i.e. with the notation that we will introduce later, a function $g_\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$) such that

$$\varphi(x) = -\sum_{\alpha\in\mathcal{A}} C^+_\alpha \log\left(|I|\{(x-l_\alpha)/|I|\}\right) - \sum_{\alpha\in\mathcal{A}} C^-_\alpha \log\left(|I|\{(r_\alpha - x)/|I|\}\right) + g_\varphi(x). \tag{2.1}$$

We refer to Figure 4 for some examples. We say that the logarithmic singularities are *of geometric type* if at least one among $C^-_{\pi_0^{-1}(d)}$ and $C^-_{\pi_1^{-1}(d)}$ is zero and at least one among $C^+_{\pi_0^{-1}(1)}$ or $C^+_{\pi_1^{-1}(1)}$ is zero (as shown in the examples in Figure 4). We denote by $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the space of functions with logarithmic singularities of geometric type. We define also the subspace $\mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \subset \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ of functions satisfying the symmetry condition

$$\sum_{\alpha\in\mathcal{A}} C^-_\alpha - \sum_{\alpha\in\mathcal{A}} C^+_\alpha = 0. \tag{2.2}$$



(a) Roof function $r \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ (b) Cocycle $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$

FIGURE 4. Examples of functions with geometric logarithmic singularities in $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$.

[18]Here $r^{(n)}(x)$ denotes the additive cocycle defined by $r^{(n)}(x) := \sum_{0\leq k<n} r(T^k x)$ if $n \geq 0$ and $r^{(n)}(x) := -\sum_{n\leq k<0} r(T^k(x))$ if $n < 0$.

2.3.2. *Special flow representations of locally Hamiltonian flows.* It is well known that locally Hamiltonian flows can be represented as special flows as follows (see for example [63, 56, 10, 20]). Consider either a minimal locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M or the restriction of a locally Hamiltonian flow on M to a minimal component $M' \subset M$. Let η be the associated closed 1-form and assume that $\eta \in \mathcal{A}$, i.e. η is Morse. Then $\psi_{\mathbb{R}}$ can be shown to be (measure theoretically) *isomorphic* to a *special flow* $T^r : I^r \to I^r$ over an interval exchange transformation $T : I \to I$ of $d \geq 1$ intervals and under a roof $r \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. The number of exchanged intervals is $d = 2g+s-1$ in the case when $\psi_{\mathbb{R}}$ is minimal and s is the number of simple saddles, or, for a minimal component M', $d = 2g' + s' - 1$, where g' is the genus of M' and s' is the number of saddles in the closure of M'. Furthermore, if $\eta \in \mathcal{U}_{min}$, the logarithmic singularities are *symmetric*, i.e. $\varphi \in \mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ (while they are *asymmetric* for special flows representations of minimal components of typical $\eta \in \mathcal{U}_{\neg min}$).

Remark 2.3. We recall for contrast that also *translation flows* can be seen as special flows over an interval exchange map, but under a roof function r which is *piecewise-constant* (and constant on each continuity interval of the IET). One can therefore see from these special representations that minimal (components of) locally Hamiltonian flows are time-changes of translation flows via a *singular* reparametrization.

2.3.3. *Reduction to skew products.* The study of (ergodic properties of) extensions can be reduced to the study of skew-products over IETs as follows.

Proposition 2.4 (Reduction of ergodicity of extensions to skew products). *Consider a Morse closed one-form $\eta \in \mathcal{A}$ on M and let $\psi_{\mathbb{R}}$ on M be the associated locally Hamiltonian flow. Consider its minimal component $M' \subset M$. For every $C^{2+\epsilon}$-map $f : M' \to \mathbb{R}$ ($\epsilon > 0$), the extension $\Phi^f_{\mathbb{R}}$ of $\psi_{\mathbb{R}}$ on M' has a Poincaré map which, in suitable coordinates, is given by a skew-product of the form*

$$(x, y) \mapsto T_{\varphi_f}(x, y) := (Tx, y + \varphi_f(x)), \qquad (x, y) \in I \times \mathbb{R}. \tag{2.3}$$

where $T = T_{(\pi,\lambda)}$ with π irreducible and the cocycle $\varphi_f : I \to \mathbb{R}$ has logarithmic singularities, i.e $\varphi_f \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, where $(I_\alpha)_{\alpha\in\mathcal{A}}$ are intervals exchanged by T.

Moreover, the extension $\Phi^f_{\mathbb{R}}$ on $M' \times \mathbb{R}$ is ergodic with respect to $\mu \times Leb$ if and only if $T_{\varphi_f} : I \times \mathbb{R} \to I \times \mathbb{R}$ is ergodic with respect to the (restriction of) the 2-dimensional Lebesgue measure on $I \times \mathbb{R}$.

We give here only a brief sketch of the proof, referring to the proof in [20] for details.

Proof. Fix a segment $\gamma \subset M' \subset M$ transverse to the flow $\psi_{\mathbb{R}}$, containing no fixed points and whose endpoints lie on outgoing separatrices of saddles. It is well known (see for example [68, Section 4.4]) that one can choose a parametrization $t \in I \to \gamma(t)$ of γ by the unit interval $I = [0, 1)$ so that the Poincaré first return map $T : I \to I$ of the flow $\psi_{\mathbb{R}}$ to γ is an IET, which is minimal by assumption. It follows that π is irreducible.

Denote by $r : I \to \mathbb{R}_{>0}$ the first return time map for the flow $(\psi_t)_{t\in\mathbb{R}}$ on M'. Then the isomorphism between the restriction of $\psi_{\mathbb{R}}$ to M' and a special flow T^r on I^r is given by

$$I^r \ni (x, r) \mapsto \psi_r(x) \in M'.$$

As recalled in the previous § 2.3.2, $r \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and moreover, if $\psi_{\mathbb{R}} \in \mathcal{U}_{min}$, i.e. $M' = M$, then $r \in \mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, see e.g. [56].

Consider now the extension $\Phi^f_{\mathbb{R}}$ of $\psi_{\mathbb{R}}$ on M' given by a bounded function $f : M' \to \mathbb{R}$. The Poincaré map of $\Phi^f_{\mathbb{R}}$ on $M' \times \mathbb{R}$ to the section $\gamma \times \mathbb{R}$ in the parametrization by $I \times \mathbb{R}$ is by construction an extension of the Poincaré map T of $\psi_{\mathbb{R}}$ to I, with return time function $r(x, y) = r(x)$ (i.e. the return time only depends on the return to I in the first coordinate, by definition of the section which has full fiber). Moreover, if we consider the cocycle

$$\varphi_f(x) := \int_0^{r(x)} f(\psi_t(x))\, dt \tag{2.4}$$

(which gives the value of the ergodic integrals of f along the trajectory from x until the first return time to the section), one can then see that the first return Poincaré map of the extension $\Phi^f_{\mathbb{R}}$ has the form (2.3). If f is a $C^{2+\epsilon}$-map, from the explicit expression (2.4) and the properties of r, one can then show that also $\varphi_f \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ (see [20] for details) and $\varphi_f \in \mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ if $\eta \in \mathcal{U}_{min}$.

The final statement is simply a consequence that ergodicity of a minimal flow is equivalent to ergodicity of its Poincaré map with respect to the induced measure, together with the remark that, under the isomorphism described above, the measure induced on the section $\gamma \times \mathbb{R}$ by the invariant measure $\mu \times Leb$ is mapped to the Lebesgue measure on $I \times \mathbb{R}$. □

The following result shows that not only ergodicity, but also reducibility of the extension $\Phi^f_{\mathbb{R}}$ can be reduced to a property of the skew product T_{φ_f} given by Proposition 2.4.

Proposition 2.5 (Reduction of reducibility to skew products, [20])**.** *For every minimal locally Hamiltonian flow $\psi_{\mathbb{R}}$ on M with non-degenerate saddles and any $f \in C^{2+\epsilon}(M)$ vanishing on $\mathrm{Fix}(\psi_{\mathbb{R}})$, the associated flow $\Phi^f_{\mathbb{R}}$ is reducible if and only if the cocycle $\varphi_f : I \to \mathbb{R}$ is a coboundary with a bounded transfer map having at least one continuity point, i.e. there exists a bounded $g : I \to \mathbb{R}$ such that $\varphi_f = g - g \circ T$ and g has at least one continuity point.*

The statement of the Proposition is proved in the proof[19] of Lemma 6.3 in [20].

3. Rauzy-Veech induction and Diophantine-type conditions

In this section we define the Diophantine-type condition on IETs which we will use to prove our main results on deviations of ergodic averages and ergodicity of extensions. The condition is described in terms of Rauzy-Veech induction, an algorithm introduced by Rauzy and Veech in [55, 64] which is now a well established tool to study IETs as well to impose Diophantine conditions on them (see e.g. [4, 5, 6, 44, 46, 63, 61, 70] and many more). We first recall some basic background material concerning Rauzy-Veech induction in § 3.1. The condition, that we call *Uniform Diophantine Condition*, or for short UDC, is defined in § 3.2 (see Definition 3 in § 3.2.2). In § 3.2.3 we also prove that this condition is satisfied by a full measure set of IETs (see Theorem 3.8).

3.1. **Rauzy-Veech induction.**

We recall here some basic definitions and notation related to Rauzy-Veech induction that will be used throughout the paper, including how it acts on Rokhlin towers (§ 3.1.4) and on Birkhoff sums (§ 3.1.5), as well as the definition of natural extension (§ 3.1.6). We recall also Oseledets theorem (§ 3.1.7).

3.1.1. *Elementary step of RV induction.* Let $T = T_{(\pi,\lambda)}$, $(\pi, \lambda) \in \mathcal{S}^0_{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}_{>0}$ be an IET satisfying Keane's condition. Then $\lambda_{\pi_0^{-1}(d)} \neq \lambda_{\pi_1^{-1}(d)}$. Let

$$\tilde{I} = \left[0, \max\left(l_{\pi_0^{-1}(d)}, l_{\pi_1^{-1}(d)}\right)\right)$$

and denote by $\mathcal{R}(T) = \tilde{T} : \tilde{I} \to \tilde{I}$ the first return map of T to the interval $\tilde{I}$. Set

$$\varepsilon(\pi, \lambda) = \begin{cases} 0 & \text{if} \quad \lambda_{\pi_0^{-1}(d)} > \lambda_{\pi_1^{-1}(d)}, \\ 1 & \text{if} \quad \lambda_{\pi_0^{-1}(d)} < \lambda_{\pi_1^{-1}(d)}. \end{cases}$$

Let us consider a pair $\tilde{\pi} = (\tilde{\pi}_0, \tilde{\pi}_1) \in \mathcal{S}^0_{\mathcal{A}}$, where

$$\begin{aligned} \tilde{\pi}_\varepsilon(\alpha) &= \pi_\varepsilon(\alpha) \text{ for all } \alpha \in \mathcal{A} \text{ and} \\ \tilde{\pi}_{1-\varepsilon}(\alpha) &= \begin{cases} \pi_{1-\varepsilon}(\alpha) & \text{if} \quad \pi_{1-\varepsilon}(\alpha) \le \pi_{1-\varepsilon} \circ \pi_\varepsilon^{-1}(d), \\ \pi_{1-\varepsilon}(\alpha) + 1 & \text{if} \quad \pi_{1-\varepsilon} \circ \pi_\varepsilon^{-1}(d) < \pi_{1-\varepsilon}(\alpha) < d, \\ \pi_{1-\varepsilon}\pi_\varepsilon^{-1}(d) + 1 & \text{if} \quad \pi_{1-\varepsilon}(\alpha) = d. \end{cases} \end{aligned}$$

As it was shown by Rauzy in [55], $\tilde{T}$ is also an IET on d-intervals

$$\tilde{T} = T_{(\tilde{\pi},\tilde{\lambda})} \text{ with } \tilde{\lambda} = A^{-1}(\pi, \lambda)\lambda, \tag{3.1}$$

where

$$A(T) = A(\pi, \lambda) = I + E_{\pi_\varepsilon^{-1}(d)\,\pi_{1-\varepsilon}^{-1}(d)} \in \mathrm{SL}(\mathbb{Z}^{\mathcal{A}}).$$

Moreover,

$$A^t(\pi, \lambda)\Omega_\pi A(\pi, \lambda) = \Omega_{\tilde{\pi}}. \tag{3.2}$$

It follows that $\ker \Omega_\pi = A(\pi, \lambda) \ker \Omega_{\tilde{\pi}}$. Thus taking $H(\pi) = \Omega_\pi(\mathbb{R}^{\mathcal{A}}) = \ker \Omega_\pi^\perp$ we get

$$H(\tilde{\pi}) = A^t(\pi, \lambda)H(\pi). \tag{3.3}$$

Moreover, $\dim H(\pi) = 2g$ and $\dim \ker \Omega_\pi = \kappa(\pi) - 1$, where $\kappa(\pi)$ is the number of singularities and g is the genus of the translation surfaces associated to π.

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[19] Note that the statement of Lemma 6.3 in [20] claims incorrectly that reducibility requires the existence of transfer function continous at *every* point, while a the *existence* of a point of continuity is sufficient. Nevertheless, the *proof* of Lemma 6.3 in [20] is correct and gives a proof of the statement of Proposition 2.5 here above.

3.1.2. *Renormalized induction.* Let $\mathcal{G}\subset\mathcal{S}^0_{\mathcal{A}}$ be any Rauzy class, i.e. a minimal subset of $\mathcal{S}^0_{\mathcal{A}}$ for which $\mathcal{G}\times\mathbb{R}^{\mathcal{A}}_{>0}$ is $\mathcal{R}$-invariant. Let

$$\Delta^{\mathcal{A}}:=\{\lambda\in\mathbb{R}^{\mathcal{A}}_{>0}:|\lambda|=1\}.$$

Then we can define the normalized Rauzy-Veech renormalization

$$\widetilde{\mathcal{R}}:\mathcal{G}\times\Delta^{\mathcal{A}}\to\mathcal{G}\times\Delta^{\mathcal{A}},\quad \widetilde{\mathcal{R}}(\pi,\lambda)=(\tilde{\pi},\tilde{\lambda}/|\tilde{\lambda}|).$$

Veech in [64] proved the existence of an $\widetilde{\mathcal{R}}$-invariant ergodic measure $\mu_{\mathcal{G}}$ ($\widetilde{\mathcal{R}}$ is recurrent with respect to $\mu_{\mathcal{G}}$) which is equivalent to the product of the counting measure on $\mathcal{G}$ and the Lebesgue measure on $\Delta^{\mathcal{A}}$.

For every T satisfying the Keane condition, the IET $\tilde{T}$ fulfills the Keane condition as well. Therefore we can iterate the renormalization procedure and generate a sequence of IETs $(\mathcal{R}^n(T))_{n\geq 0}$. For every $n\geq 1$ let

$$A^{(n)}(T)=A(T)\cdot A(\mathcal{R}(T))\cdot\ldots\cdot A(\mathcal{R}^{n-1}(T)).$$

In what follows, the norm of a vector is defined as the sum of the absolute value of coefficients and for any matrix $B=[B_{\alpha\beta}]_{\alpha,\beta\in\mathcal{A}}$ we set $\|B\|=\max_{\alpha\in\mathcal{A}}\sum_{\beta\in\mathcal{A}}|B_{\alpha\beta}|$.

3.1.3. *Accelerations.* Let $T:I\to I$ be an arbitrary IET satisfying Keane's condition. Let $(n_k)_{k\geq 0}$ be an increasing sequence of integer numbers with $n_0=0$, called an *accelerating sequence.* For every $k\geq 0$ let $T^{(k)}:=\mathcal{R}^{n_k}(T):I^{(k)}\to I^{(k)}$. Denote by $(\pi^{(k)},\lambda^{(k)})$ the pair defining $T^{(k)}$ and by $\lambda^{(k)}=(\lambda^{(k)}_\alpha)_{\alpha\in\mathcal{A}}=(|I^{(k)}_\alpha|)_{\alpha\in\mathcal{A}}$ the vector which determines $T^{(k)}$.
In view of (3.1), letting $Z(k+1):=A^{(n_{k+1}-n_k)}(\mathcal{R}^{n_k}(T))^t$ for $k\geq 0$ we have

$$\lambda^{(k)}=Z(k+1)^t\lambda^{(k+1)}\text{ for all }k\geq 0.$$

We use the notation from [44], but adopt the convention later introduced in [48]. For each $0\leq k<l$ let

$$Q(k,l)=Z(l)\cdot Z(l-1)\cdot\ldots\cdot Z(k+2)\cdot Z(k+1)=A^{(n_l-n_k)}(\mathcal{R}^{n_k}(T))^t.$$

Then $Q(k,l)\in SL_{\mathcal{A}}(\mathbb{Z})$ and

$$\lambda^{(k)}=Q(k,l)^t\lambda^{(l)}.$$

It follows that

$$|I^{(k)}|\leq|I^{(l)}|\|Q(k,l)\|. \tag{3.4}$$

We will write $Q(k)$ for $Q(0,k)$.
We say that $Z(k)$, $k\in\mathbb{N}$ (resp. $Q(k,l)$) are the *matrices* (resp. the *product matrices*) of the acceleration of A along the (accelerating) sequence $(n_k)_{k\in\mathbb{N}}$

3.1.4. *Rokhlin towers.* By definition, $T^{(l)}:I^{(l)}\to I^{(l)}$ is the first return map of $T^{(k)}:I^{(k)}\to I^{(k)}$ to the interval $I^{(l)}\subset I^{(k)}$. Moreover, $Q_{\alpha\beta}(k,l)$ is the time spent by any point of $I^{(l)}_\alpha$ in $I^{(k)}_\beta$ until it returns to $I^{(l)}$. It follows that

$$Q_\alpha(k,l)=\sum_{\beta\in\mathcal{A}}Q_{\alpha\beta}(k,l)$$

is the first return time of points of $I^{(l)}_\alpha$ to $I^{(l)}$.
The map $T^{(k)}:I^{(k)}\to I^{(k)}$ can be then represented as a *Rokhlin skyscraper* as follows. For every $\alpha\in\mathcal{A}$, we say that the set

$$\{(T^{(k)})^i(I^{(l)}_\alpha),\qquad 0\leq i<Q_\alpha(k,l)\}$$

is called a *Rokhlin tower.* Notice that the $Q_\alpha(k,l)$ sets part of it are *disjoint* intervals called *floors* of the tower and that, for $0\leq i<Q_\alpha(k,l)$, $T^{(k)}$ acts on the i^{th} floor $(T^{(k)})^i(I^{(l)}_\alpha)$ mapping it to the $(i+1)^{th}$ one. The union of all Rokhlin towers over $\alpha\in\mathcal{A}$ gives $I^{(k)}$.

3.1.5. *Special Birkhoff sums.* We deal with the *special Birkhoff sums* operators $S(k,l):L^1(I^{(k)})\to L^1(I^{(l)})$ for $0\leq k<l$ defined by

$$S(k,l)f(x)=\sum_{0\leq j<Q_\alpha(k,l)}f((T^{(k)})^jx)\quad\text{ if }\quad x\in I^{(l)}_\alpha.$$

Let $T=T^{(0)}$ be an IET satisfying Keane's condition. For every $k\geq 0$ let $\Gamma^{(k)}\subset L^1(I^{(k)})$ be the subspace of functions on $I^{(k)}$ which are constant on each $I^{(k)}_\alpha$, $\alpha\in\mathcal{A}$. Then for $0\leq k<l$ we have $S(k,l)\Gamma^{(k)}=\Gamma^{(l)}$. Let us identify every function $\sum_{\alpha\in\mathcal{A}}h_\alpha\chi_{I^{(k)}_\alpha}\in\Gamma^{(k)}$ with the vector $h=(h_\alpha)_{\alpha\in\mathcal{A}}\in\mathbb{R}^{\mathcal{A}}$. Clearly $\Gamma^{(k)}$ is isomorphic to $\mathbb{R}^{\mathcal{A}}$. Under the identification, the operator $S(k,l)$ is the linear automorphism of $\mathbb{R}^{\mathcal{A}}$ whose matrix in the canonical basis is $Q(k,l)$. In view of (3.3) for $0\leq k<l$ we have

$$Q(k,l)H(\pi^{(k)})=H(\pi^{(l)}).$$

For every $k \geq 0$ let

$$\Gamma_s^{(k)} := \{h \in \Gamma^{(k)} : \exists_{\sigma>0}\exists_{C>0}\forall_{l>k}\, \|Q(k,l)h\| \leq C\|Q(k,l)\|^{-\sigma}\}.$$

The space $\Gamma_s^{(k)}$ is a subspace of $H(\pi^{(k)})$ and for every $l > k$ we have

$$Q(k,l)\Gamma_s^{(k)} = \Gamma_s^{(l)}.$$

Therefore, the restriction operator and the quotient operators of $Q(k,l)$

$$Q_s(k,l) : \Gamma_s^{(k)} \to \Gamma_s^{(l)}, \quad Q_\flat(k,l) : \Gamma^{(k)}/\Gamma_s^{(k)} \to \Gamma^{(l)}/\Gamma_s^{(l)}, \quad Q_\sharp(k,l) : H(\pi^{(k)})/\Gamma_s^{(k)} \to H(\pi^{(l)})/\Gamma_s^{(l)}$$

are well defined and are invertible. Arguments presented in Section 3.2 in [48] shows that if $\dim \Gamma_s^{(0)} = g$ then

$$\|Q_\sharp(k,l)^{-1}\| = \|Q_s(k,l)\|. \tag{3.5}$$

3.1.6. *The natural extension.* Rauzy-Veech induction is not intertible, but it can be extended to an invertible induction on the space of *zippered rectangles* (as described in the seminar paper by Veech [64]). We recall briefly the construction. We refer the reader who needs more background to the lecture notes by Yoccoz [68] or Viana [66].

For every $\pi \in S_0^{\mathcal{A}}$ let

$$\Theta_\pi := \Big\{\tau \in \mathbb{R}^{\mathcal{A}} : \sum_{\pi_0(\alpha)\leq k} \tau_\alpha > 0, \sum_{\pi_1(\alpha)\leq k} \tau_\alpha < 0 \text{ for } 1 \leq k < d\Big\}.$$

For every $\tau \in \Theta_\pi$ let $h = h(\tau) = \Omega\tau \in \mathbb{R}^{\mathcal{A}}_{>0}$. For every Rauzy class $\mathcal{G} \subset S_0^{\mathcal{A}}$ let

$$X(\mathcal{G}) = \bigcup_{\pi\in\mathcal{G}} \{(\pi,\lambda,\tau) \in \{\pi\} \times \Delta^{\mathcal{A}} \times \Theta_\pi : \langle\lambda, \Omega_\pi\tau\rangle = 1\}. \tag{3.6}$$

For every $(\pi,\lambda,\tau) \in X(\mathcal{G})$ denote by $M(\pi,\lambda,\tau)$ the translation surface arising in the zippered rectangles process. Then $M(\pi,\lambda,\tau)$ is zippered from the rectangles $I_\alpha \times [0,h_\alpha]$, $\alpha \in \mathcal{A}$ such that the points $\sum_{\pi_0(\alpha)\leq k}(\lambda_\alpha + i\tau_\alpha)$, $0 \leq k \leq d$ are its singular points. Moreover, the IET T is the first return map to $I \subset M(\pi,\lambda,\tau)$ for the vertical flow on $M(\pi,\lambda,\tau)$.

The map $\widehat{\mathcal{R}} : X(\mathcal{G}) \to X(\mathcal{G})$ given by

$$\widehat{\mathcal{R}}(\pi,\lambda,\tau) = \Big(\tilde{\pi}, \frac{A^{-1}(\pi,\lambda)\lambda}{|A^{-1}(\pi,\lambda)\lambda|}, |A^{-1}(\pi,\lambda)\lambda| A^{-1}(\pi,\lambda)\tau\Big)$$

is an invertible map and is the natural extension of $\widetilde{\mathcal{R}}$. Denote by $\widehat{\mu}_{\mathcal{G}}$ the natural extension of the measure $\mu_{\mathcal{G}}$. Then $\widehat{\mu}_{\mathcal{G}}$ is $\widehat{\mathcal{R}}$-invariant and $\widehat{\mathcal{R}}$ is recurrent and ergodic with respect to $\widehat{\mu}_{\mathcal{G}}$.

3.1.7. *Oseledets splitting.* Let us extend the cocycle $A : \mathcal{G} \times \Lambda^{\mathcal{A}} \to SL_{\mathcal{A}}(\mathbb{Z})$ to $\widehat{A} : X(\mathcal{G}) \to SL_{\mathcal{A}}(\mathbb{Z})$ by

$$\widehat{A}(\pi,\lambda,\tau) := A(\lambda,\tau)$$

and let us consider the cocycle $\widehat{A} : \mathbb{Z} \times X(\mathcal{G}) \to SL_{\mathcal{A}}(\mathbb{Z})$

$$\widehat{A}^{(n)}(\pi,\lambda,\tau) = \begin{cases} \widehat{A}(\pi,\lambda,\tau) \cdot \widehat{A}(\widehat{\mathcal{R}}(\pi,\lambda,\tau)) \cdot \ldots \cdot \widehat{A}(\widehat{\mathcal{R}}^{n-1}(\pi,\lambda,\tau)) & \text{if } n \geq 0 \\ \widehat{A}(\widehat{\mathcal{R}}^{-1}(\pi,\lambda,\tau)) \cdot \widehat{A}(\widehat{\mathcal{R}}^{-2}(\pi,\lambda,\tau)) \cdot \ldots \cdot \widehat{A}(\widehat{\mathcal{R}}^{n}(\pi,\lambda,\tau)) & \text{if } n < 0. \end{cases}$$

Then

$$\widehat{A}^{(n)}(\pi,\lambda,\tau) = A^{(n)}(\pi,\lambda) \text{ if } n \geq 0. \tag{3.7}$$

Let $Y \subset X(\mathcal{G})$ be a subset with $0 < \widehat{\mu}_{\mathcal{G}}(Y) < +\infty$. For a.e. $(\pi,\lambda,\tau) \in Y$ let $r(\pi,\lambda,\tau) \geq 1$ by the first return time of (π,λ,τ) for the map $\widehat{\mathcal{R}}$. Denote by $\widehat{\mathcal{R}}_Y : Y \to Y$ the induced map and by $\widehat{A}_Y : Y \to SL_{\mathcal{A}}(\mathbb{Z})$ the induced cocycle, i.e.

$$\widehat{\mathcal{R}}_Y(\pi,\lambda,\tau) = \widehat{\mathcal{R}}^{r(\pi,\lambda,\tau)}(\pi,\lambda,\tau), \quad \widehat{A}_Y(\pi,\lambda,\tau) = \widehat{A}^{(r(\pi,\lambda,\tau))}(\pi,\lambda,\tau)$$

for a.e. $(\pi,\lambda,\tau) \in Y$. Let $\widehat{\mu}_Y$ be the restriction of $\widehat{\mu}_{\mathcal{G}}$ to Y. Then $\widehat{\mathcal{R}}_Y$ is an ergodic measure-preserving invertible map on $(Y, \widehat{\mu}_Y)$.

Suppose that $\log\|\widehat{A}_Y\|$ and $\log\|\widehat{A}_Y^{-1}\|$ are integrable. Then, by Oseledets theorem, symplecticity of $\widehat{A}_Y$ (see [70]) and simplicity of spectrum (see [5]), there exists $\lambda_1 > \ldots > \lambda_g > 0$ such that for a.e. $(\pi,\lambda,\tau) \in Y$ we have a *Oseledets splitting*

$$\mathbb{R}^{\mathcal{A}} = \bigoplus_{-g\leq i\leq g} \Gamma_i(\pi,\lambda,\tau)$$

for which

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|\widehat{A}_Y^{(n)}(\pi,\lambda,\tau)^t v\| = \lambda_i \text{ if } v\in\Gamma_i(\pi,\lambda,\tau) \text{ and } i>0$$
$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|\widehat{A}_Y^{(n)}(\pi,\lambda,\tau)^t v\| = -\lambda_i \text{ if } v\in\Gamma_i(\pi,\lambda,\tau) \text{ and } i<0$$
$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|\widehat{A}_Y^{(n)}(\pi,\lambda,\tau)^t v\| = 0 \text{ if } v\in\Gamma_0(\pi,\lambda,\tau),$$

$$\dim\Gamma_i(\pi,\lambda,\tau)=1 \text{ if } i\neq 0, \quad \dim\Gamma_0(\pi,\lambda,\tau)=\kappa-1.$$

Furthermore, we have that

$$H(\pi)=\bigoplus_{i\neq 0}\Gamma_i(\pi,\lambda,\tau).$$

We denote by $\Gamma_s(\pi,\lambda,\tau)$ and $\Gamma_u(\pi,\lambda,\tau)$ the *stable* and *unstable* spaces, which are given respectively by

$$\Gamma_s(\pi,\lambda,\tau):=\bigoplus_{-g\le i\le -1}\Gamma_i(\pi,\lambda,\tau) \text{ and } \Gamma_u(\pi,\lambda,\tau):=\bigoplus_{1\le i\le g}\Gamma_i(\pi,\lambda,\tau). \tag{3.8}$$

Notice that both $\Gamma_s(\pi,\lambda,\tau)$ and $\Gamma_u(\pi,\lambda,\tau)$ have exactly dimension g. We say in this case that the Oseledets splitting is *of hyperbolic type*.

3.1.8. *Veech bases for the kernel* $\ker\Omega_\pi$*.* In [64, 65], Veech explicitly defines a bases for $\ker\Omega_\pi$ for every π in a given Rauzy class. We recall the construction (which uses the classical notation for the permutation describing the IETs, also called *monodromy*, namely the permutation $\pi_1\circ\pi_0^{-1}$). Let us first define the *extended permutation* $p:\{0,1,\dots,d,d+1\}\to\{0,1,\dots,d,d+1\}$ to be the permutation

$$p(j)=\begin{cases}\pi_1\circ\pi_0^{-1}(j) & \text{if} \quad 1\le j\le d\\ j & \text{if} \quad j=0,d+1.\end{cases}$$

Following Veech (see [64, 65]), denote by $\sigma=\sigma_\pi$ the corresponding permutation on $\{0,1,\dots,d\}$,

$$\sigma(j)=p^{-1}(p(j)+1)-1 \text{ for } 0\le j\le d.$$

Notice that (recalling Remark 2.2 and the definition just before of $\widehat{T}$), we have $\widehat{T}_{(\pi,\lambda)}r_{\pi_0^{-1}(j)}=T_{(\pi,\lambda)}r_{\pi_0^{-1}(\sigma j)}$ for all $j\neq 0,p^{-1}(d)$.

Denote by $\Sigma(\pi)$ the set of orbits for the permutation σ. Let $\Sigma_0(\pi)$ stand for the subset of orbits that do not contain zero. Then $\Sigma(\pi)$ corresponds to the set of singular points of any translation surface associated to π and hence $\#\Sigma(\pi)=\kappa(\pi)$.

For every $\mathcal{O}\in\Sigma(\pi)$ denote by $b(\mathcal{O})\in\mathbb{R}^{\mathcal{A}}$ the vector given by

$$b(\mathcal{O})_\alpha=\chi_{\mathcal{O}}(\pi_0(\alpha))-\chi_{\mathcal{O}}(\pi_0(\alpha)-1) \text{ for } \alpha\in\mathcal{A}, \tag{3.9}$$

where $\chi_{\mathcal{O}}(j)=1$ iff $j\in\mathcal{O}$ and 0 otherwise. Moreover, for every $\mathcal{O}\in\Sigma(\pi)$, we denote by

$$\mathcal{A}_{\mathcal{O}}^-=\{\alpha\in\mathcal{A},\ \pi_0(\alpha)\in\mathcal{O}\}, \qquad \mathcal{A}_{\mathcal{O}}^+=\{\alpha\in\mathcal{A},\ \pi_0(\alpha)-1\in\mathcal{O}\}. \tag{3.10}$$

If $\alpha\in\mathcal{A}_{\mathcal{O}}^+$ (respectively $\alpha\in\mathcal{A}_{\mathcal{O}}^-$) then the left (respectively right) endpoint of I_α belongs to a separatrix of the saddle represented by $\mathcal{O}$.

Lemma 3.1 (see [65])**.** *For every irreducible pair* π *we have:*

(i) $\sum_{\mathcal{O}\in\Sigma(\pi)}b(\mathcal{O})=0$*;*
(ii) the vectors $b(\mathcal{O})$*,* $\mathcal{O}\in\Sigma_0(\pi)$ *are linearly independent;*
(iii) the linear subspace generated by $\{b(\mathcal{O}),\ \mathcal{O}\in\Sigma_0(\pi)\}$ *is equal to* $\ker\Omega_\pi$*.*

Moreover, $h\in H(\pi)$ *if and only if* $\langle h,b(\mathcal{O})\rangle=0$ *for every* $\mathcal{O}\in\Sigma(\pi)$*.*

Veech also describes how these bases change under Rauzy-Veech induction:

Lemma 3.2 (see Veech, [65])**.** *Suppose that* $T_{(\tilde\pi,\tilde\lambda)}=\mathcal{R}(T_{(\pi,\lambda)})$*. Then there exists a bijection* $\xi:\Sigma(\pi)\to\Sigma(\tilde\pi)$ *such that*

$$A(\pi,\lambda)^{-1}b(\mathcal{O})=b(\xi\mathcal{O}), \qquad \textit{for all } \ \mathcal{O}\in\Sigma(\pi).$$

3.1.9. *The boundary operator.* The following operator ∂_π is known by *boundary operator* (as a special case of the more general operator introduced in [44], see § 4.1.3). Let $\Sigma(\pi)$ and $\mathcal{A}^{\pm}_{\mathcal{O}}$ be as in the previous subsection.

Definition 1. Let $\partial_\pi : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\Sigma(\pi)}$ stand for the linear transformation which maps a vector $h \in \mathbb{R}^{\mathcal{A}}$ to the vector in $\mathbb{R}^{\Sigma(\pi)}$ whose coordinates $(\partial_\pi h)_{\mathcal{O}}, \mathcal{O} \in \Sigma(\pi)$ are given by

$$(\partial_\pi h)_{\mathcal{O}} := \langle h, b(\mathcal{O})\rangle = \sum_{\alpha\in\mathcal{A}^-_{\mathcal{O}}} h_\alpha - \sum_{\alpha\in\mathcal{A}^+_{\mathcal{O}}} h_\alpha, \qquad \text{for } \mathcal{O} \in \Sigma(\pi).$$

One sees (in light of Remark 2.1) that the image of ∂_π is:

$$\partial_\pi(\mathbb{R}^{\mathcal{A}}) = \Big\{(x_{\mathcal{O}})_{\mathcal{O}\in\Sigma(\pi)} : \sum_{\mathcal{O}\in\Sigma(\pi)} x_{\mathcal{O}} = 0\Big\}. \tag{3.11}$$

Remark 3.3. We can identify a vector $h \in \mathbb{R}^{\mathcal{A}}$ with a piecewise constant function g_h, which gives the constant value h_α to the subinterval I_α. Then the operator ∂ can be thought of as acting on piecewise constant functions and producing, as a value at $\mathcal{O} \in \Sigma(\pi)$, the *sum of jumps* of the function g_h at the endpoints corresponding to the singularity labelled by $\mathcal{O}$.

Two extensions of this operator (viewed as in the previous remark as an operator on functions) will be defined later, to functions piecewise absolutely continuous on each I_α (§ 4.1.3) and to functions with logarithmic singularities (§ 4.3.3).

3.1.10. *Boundary operator estimate.* Let $H(\pi) := \ker \partial_\pi$. Denote by $p_{H(\pi)} : \mathbb{R}^{\mathcal{A}} \to H(\pi)$ the orthogonal projection on $H(\pi)$ with respect to the standard scalar product on $\mathbb{R}^{\mathcal{A}}$.

Lemma 3.4. *For any $h \in \mathbb{R}^{\mathcal{A}}$, we have*

$$\|p_{H(\pi)}h\| \le \sqrt{d}\|h\|. \tag{3.12}$$

Moreover, for any Rauzy class $\mathcal{G} \subset \mathcal{S}^0_A$ there exists a positive constant $C_{\mathcal{G}}$ such that for every $\pi \in \mathcal{G}$ and $h \in \mathbb{R}^{\mathcal{A}}$ we have

$$\|h - p_{H(\pi)}h\| \le C_{\mathcal{G}}\|\partial_\pi h\|. \tag{3.13}$$

Proof. Let $H(\pi)^\perp \subset \mathbb{R}^{\mathcal{A}}$ be the orthogonal complement of $H(\pi)$. By Lemma 3.1, $\partial_\pi : H(\pi)^\perp \to \mathbb{R}^{\Sigma(\pi)}$ is a linear isomorphism. It follows that there exists $C_\pi > 0$ such that

$$\|h\| \le C_\pi\|\partial_\pi h\| \quad \text{for all} \quad h \in H(\pi)^\perp.$$

Hence (3.13) holds with $C_{\mathcal{G}} = \max\{C_\pi : \pi \in \mathcal{G}\}$. Denote by $\|\cdot\|_2$ the Euclidean norm on $\mathbb{R}^{\mathcal{A}}$. Since $\|h\|_2 \le \|h\| \le \sqrt{d}\|h\|_2$ and $p_{H(\pi)}$ is an orthogonal projection, we have

$$\|p_{H(\pi)}h\| \le \sqrt{d}\|p_{H(\pi)}h\|_2 \le \sqrt{d}\|h\|_2 \le \sqrt{d}\|h\|.$$

□

3.2. The Uniform Diophantine-type Condition and its full measure.

We will now define the Diophantine-type condition that we will use. First, it is convenient to introduce an acceleration of Rauzy-Veech induction which produces times which we call *Rokhlin-balanced.* We then define the condition and prove that it has full measure.

3.2.1. *The Rokhlin-balanced acceleration.* The following acceleration of Rauzy-Veech induction produces times of the Rauzy-Veech algorithm where the corresponding Rokhlin towers (see 3.1.4) are *balanced* in the sense that all bases have comparable lengths (see (B1) in Definition 2) and all the towers travel together for a long enough time (see (B2) in Definition 2). We call these times *Rokhlin-balanced.*

Definition 2 (Rokhlin-balance). Let us say that an accelerating sequence $(n_k)_{k\ge 0}$ is *Rokhlin-balanced* if there exist constants $\kappa > 1$ and $0 < \delta < 1$ such that the following two conditions hold for every $k \in \mathbb{N}$:

(B1) $|I^{(k)}| \le \kappa\, |I^{(k)}_\alpha|$ for all $k \ge 1$ and $\alpha \in \mathcal{A}$;

(B2) for every $k \ge 1$ there exists a natural number $0 < p_k \le \min_{\alpha\in\mathcal{A}} Q_\alpha(k)$ such that

$\{T^i I^{(k)} : 0 \le i < p_k\}$ is a Rokhlin tower of intervals with measure greater than $\delta|I|$.

We say that an IET is *Rokhlin-balanced* if it satisfies Keane's condition and it admits a Rokhlin balanced accelerating sequence $(n_k)_{k\ge 0}$.

Remark 3.5. Notice that by conditions (B1) and (B2), for every $\alpha \in \mathcal{A}$ and $k \geq 1$ we have

$$\|Q(k)\||I^{(k)}| \leq \kappa \sum_{\alpha\in\mathcal{A}} Q_\alpha(k)|I^{(k)}_\alpha| = \kappa|I|, \text{ and} \tag{3.14}$$

$$Q_\alpha(k)\lambda^{(k)}_\alpha \geq \frac{1}{\kappa}p_k|I^{(k)}| \geq \frac{\delta}{\kappa}|I|. \tag{3.15}$$

so that each Rokhlin tower of a balanced acceleration induction time has measure uniformly bounded below.

Let us show that for almost every IET one can find a Rokhlin-balanced sequence by considering returns of Rauzy-Veech induction to special compact sets (for the parameter space of the natural extension, see § 3.1.6). Let us recall that $X(\mathcal{G})$ denotes the domain of the natural extension of the Rauzy-Veech induction (see (3.6) in § 3.1.6).

Lemma 3.6. *Let π be irreducible. For Lebesgue-almost every choice of λ, the IET $T = T_{(\pi,\lambda)}$ is Rokhlin-balanced. Furthermore, for every $0 < \delta < 1$ one can define a set $Y = Y(\delta) \subset X(\mathcal{G})$ such that a Rokhlin-balanced accelerating sequence with constant δ is given by returns of the natural extension of Rauzy-Veech induction to Y.*

Proof. Fix $0 < \delta < 1$. Let us consider a subset $Y = Y(\delta) \subset X(\mathcal{G})$ which satisfies:

(i) its projection Y_0 on $\mathcal{G} \times \Lambda^{\mathcal{A}}$ is precompact with respect to the Hilbert metric;
(ii) for every $(\pi, \lambda, \tau) \in Y$ we have

$$\min\Big\{\Big\{\sum_{\pi_0(\alpha)\leq k} \tau_\alpha : 1 \leq k < d\Big\} \cup \{h_\alpha(\tau) : \alpha \in \mathcal{A}\}\Big\} > \delta \max\{h_\alpha(\tau) : \alpha \in \mathcal{A}\};$$

Let $R > 0$ be such that $Y_0 \subset \mathcal{G} \times \overline{B}_H((1/d, \dots, 1/d), R)$, where $\overline{B}_H((1/d, \dots, 1/d), R)$ is the closed ball (with respect to the Hilbert metric d_H) of radius R and center at the center of the simplex $\Lambda^{\mathcal{A}}$.

Balance at visit times. Consider any sequence $(n_k)_{k\geq 1}$ which corresponds to visits to the set Y. By definition, for every k belonging to this subsequence, $(\pi^{(k)}, \lambda^{(k)}, \tau^{(k)}) \in Y$. It follows that $d_H\big(\lambda^{(k)}, (1/d, \dots, 1/d)\big) \leq R$. Therefore

$$\max_{\alpha\in\mathcal{A}} |I^{(k)}_\alpha| / \min_{\alpha\in\mathcal{A}} |I^{(k)}_\alpha| \leq e^R,$$

which implies the condition (B1) for $\kappa := e^R$.
As $(\pi^{(k)}, \lambda^{(k)}, \tau^{(k)}) = \widehat{\mathcal{R}}^{n_k}(\pi, \lambda, \tau)) \in Y$, by condition (ii) in the choice of Y, taking

$$t^{(k)} := \min\Big\{\Big\{\sum_{\pi^{(k)}_0(\alpha)\leq l} \tau^{(k)}_\alpha : 1 \leq l < d\Big\} \cup \{h^{(k)}_\alpha : \alpha \in \mathcal{A}\}\Big\} \quad (h^{(k)} = h(\tau^{(k)}))$$

we have that $I^{(k)} \times [0, t^{(k)}]$ is a rectangle (without singular points inside) in the translation surface $M(\pi^{(k)}, \lambda^{(k)}, \tau^{(k)})$ $(= M(\pi, \lambda, \tau))$ and its area is greater than

$$t^{(k)} \sum_{\alpha\in\mathcal{A}} \lambda^{(k)}_\alpha > \delta \max_{\alpha\in\mathcal{A}} h^{(k)}_\alpha \sum_{\alpha\in\mathcal{A}} \lambda^{(k)}_\alpha \geq \delta\langle\lambda^{(k)}, h^{(k)}\rangle = \delta|I|.$$

This gives (B2) with $p_k := [t^{(k)}/\max_{\alpha\in\mathcal{A}} h_\alpha(\tau)]$ and $\delta := \frac{\delta^2}{2} < \frac{\delta}{2}\frac{\min_{\alpha\in\mathcal{A}} h_\alpha(\tau)}{\max_{\alpha\in\mathcal{A}} h_\alpha(\tau)}$.

Typical Rokhlin balance. It now follows from Poincaré recurrence theorem (and absolute continuity and finiteness of the Veech invariant measure, see [64]) that almost every IET visits $Y(\delta)$ infinitely often and hence is Rokhlin-balanced. □

3.2.2. *The Uniform Diophantine Condition definition.* The Diophantine-type condition that we will use in the main theorems is the following.

Definition 3 (UDC). An IET $T : I \to I$ satisfying Keane's condition, satisfies the *Uniform Diophantine Condition* UDC if T is Rokhlin-balanced (in the sense of Definition 2), and for every $\tau > 0$ there exist constants $0 < c < C$, a Rokhlin-balanced accelerating sequence $(n_k)_{k\geq 0}$ and an increasing sequence of integers $(r_n)_{n\geq 0}$ with $r_0 = 0$ and $r_n/n \to \alpha > 0$, so that:

(O) T is *Oseledets generic*, i.e. there exists an extension (π, λ, τ) of $T = T_{(\pi,\lambda)}$ such that it admits an Oseledets splitting of hyperbolic type, as in § 3.1.7;

and, furthermore, the matrices $Z(k)$ and product matrices $Q(k, l)$ of the acceleration along the subsequence $(n_k)_{k\in\mathbb{N}}$ (see § 3.1.3) satisfy the following conditions:

(UDC1) $\|Q_s(k, l)\| \leq Ce^{-\lambda(l-k)}$ for all $0 \leq k \leq l$, where $\lambda = \lambda_g/2$;

(UDC2) $\|Z(k + 1)\| \leq Ce^{\tau|k-r_n|}$ for all $k \geq 0$ and $n \geq 0$;

(UDC3) $ce^{\lambda_1 k} \leq \|Q(k)\| \leq Ce^{\lambda_1(1+\tau)k}$ for all $k \geq 0$;

Remark 3.7. By conditions (UDC2) and (UDC3), there exists $C' > 0$ such that

$$\|Z(k+1)\| = O(\|Q(k)\|^{\tau}). \tag{3.16}$$

Then using arguments from Section 1.3.1 in [44], one can show that

$$\|Q(k)\| = O(\min_{\alpha\in\mathcal{A}} Q_\alpha(k)^{1+\tau}). \tag{3.17}$$

Thus, the UDC condition implies condition (a) of the Roth-type Diophantine condition defined in [44]. The other two conditions (as well as the last assumption of the *restricted* Roth-type condition[20]) also hold, in view of the Oseledets genericity assumption (O) (see for example Remark 3.4 in [48]). Thus IETs which satisfy the UDC are in particular of (restricted) Roth-type.

3.2.3. *Full measure of the UDC.* Let us show that the UDC condition has full measure.

Theorem 3.8. *Almost every IET satisfies the UDC Diophantine condition.*

Proof. We split the proof in several steps.

Construction of a good recurrence set. Let us consider a subset $Y \subset X(\mathcal{G})$ which satisfies the assumptions (i) and (ii) in the proof of Lemma 3.6, which guarantees that visits to Y give a Rokhlin-balanced sequence, and furthermore such that:

(iii) $\widehat{\mu}(Y)$ is finite, so $\widehat{\mu}_Y := \widehat{\mu}/\widehat{\mu}(Y)$ is a probability measure;

(iv) the functions $\log\|\widehat{A}_Y\|$ and $\log\|\widehat{A}_Y^{-1}\|$ are integrable with respect to $\widehat{\mu}_Y$.

Let $\lambda_1 > \ldots > \lambda_g > 0$ the positive Lyapunov exponents of the corresponding accelerated cocycle, which are g and distinct in view of [22] and [5]. Let $\lambda := \lambda_g/2$ and $\kappa = de^R$. Fix $0 < \tau < \lambda_g/2$. Since for $\widehat{\mu}_Y$-a.e. $(\pi,\lambda,\tau)\in Y$ we have

$$\lim_{n\to+\infty}\frac{1}{n}\log\|\widehat{A}_Y^{(n)}(\pi,\lambda,\tau)^t\restriction_{\Gamma_s(\pi,\lambda,\tau)}\| = -\lambda_g,$$

the map from Y to $\mathbb{R}$ given by

$$(\pi,\lambda,\tau)\mapsto \sup_{n\geq 0} e^{(\lambda_g-\tau)n}\|\widehat{A}_Y^{(n)}(\pi,\lambda,\tau)^t\restriction_{\Gamma_s(\pi,\lambda,\tau)}\|$$

is a.e. defined and measurable. Therefore, there exists a subset $K\subset Y$ with $\widehat{\mu}_Y(K)/\widehat{\mu}_Y(Y) > 1-\tau/2$ and a constant $C>0$ such that if $(\pi,\lambda,\tau)\in K$ then for every $n\geq 0$ we have

$$\|\widehat{A}_Y^{(n)}(\pi,\lambda,\tau)^t\restriction_{\Gamma_s(\pi,\lambda,\tau)}\| \leq Ce^{-(\lambda_g-\tau)n}\leq Ce^{-\lambda n}. \tag{3.18}$$

First acceleration. Let us consider the induced map $\widehat{\mathcal{R}}_K : K\to K$ and the induced cocycle $\widehat{A}_K : K\to SL_{\mathcal{A}}(\mathbb{Z})$. Then $\widehat{\mathcal{R}}_K(\pi,\lambda,\tau) = \widehat{\mathcal{R}}_Y^{r_K(\pi,\lambda,\tau)}(\pi,\lambda,\tau)$, where $r_K(\pi,\lambda,\tau)\geq 1$ is the first return time of $(\pi,\lambda,\tau)\in K$ to K for the map $\widehat{\mathcal{R}}_Y$. Let $r_K^{(n)} := \sum_{0\leq i<n} r_k\circ\widehat{\mathcal{R}}_K$ for every $n\geq 0$. Then

$$\frac{r_K^{(n)}}{n}\to\frac{\widehat{\mu}_Y(Y)}{\widehat{\mu}_Y(K)} \text{ a.e. on } K$$

and furthermore

$$\widehat{A}_K^{(n)} = \widehat{A}_Y^{(r_K^{(n)})} \text{ for every } n\geq 0.$$

In view of (3.18), for every $(\pi,\lambda,\tau)\in K$ we have

$$\|\widehat{A}_K^{(n)}(\pi,\lambda,\tau)^t\restriction_{\Gamma_s(\pi,\lambda,\tau)}\|\leq Ce^{-\lambda r_K^{(n)}(\pi,\lambda,\tau)}\leq Ce^{-\lambda n} \tag{3.19}$$

and for a.e. $(\pi,\lambda,\tau)\in K$ we have

$$\lim_{n\to+\infty}\frac{1}{n}\log\|\widehat{A}_K^{(n)}(\pi,\lambda,\tau)\| = \lambda_1\frac{\widehat{\mu}_Y(Y)}{\widehat{\mu}_Y(K)}\in(\lambda_1,\lambda_1(1+\tau)). \tag{3.20}$$

Second acceleration. Since the functions $\log\|\widehat{A}_K\|$ and $\log\|\widehat{A}_K^{-1}\|$ are integrable, for a.e. $(\pi,\lambda,\tau)\in K$ we have $\log\|\widehat{A}_K(\widehat{\mathcal{R}}_K^n(\pi,\lambda,\tau))\|/n\to 0$ as $|n|\to+\infty$, also the map from K to $\mathbb{R}$ given by

$$(\pi,\lambda,\tau)\mapsto\sup_{n\in\mathbb{Z}} e^{-\tau|n|}\|\widehat{A}_K(\widehat{\mathcal{R}}_K^n(\pi,\lambda,\tau))\|$$

is a.e. defined and measurable. Therefore, there exists a subset $K'\subset K$ with $\widehat{\mu}_K(K')>0$ and a constant $C'>0$ such that if $(\pi,\lambda,\tau)\in K'$ then for every $n\in\mathbb{Z}$ we have

$$\|\widehat{A}_K(\widehat{\mathcal{R}}_K^n(\pi,\lambda,\tau))\|\leq C'e^{\tau|n|}. \tag{3.21}$$

[20]In [46], Marmi, Moussa and Yoccoz introduced a more restrictive (but still full measure) Diophantine-type condition, that they called *restricted Roth-type*: in addition to all the properties of *Roth-type*, one requests in this case that the stable space has exactly dimension g. This holds for IETs which satisfy the UDC in view of the Oseledets genericity assumption (O), since we require that the splitting is of hyperbolic type, which means exactly that there are g positive exponents, see § 3.1.7).

Moreover, for a.e. $(\pi,\lambda,\tau)\in K'$ there exists an increasing sequence of non-negative integer numbers $(r_n(\pi,\lambda,\tau))_{n\geq 1}$ such that $r_1(\pi,\lambda,\tau)=0$ and

$$\widehat{\mathcal{R}}_K^{r_n(\pi,\lambda,\tau)}(\pi,\lambda,\tau)\in K' \text{ for all } n\geq 0 \text{ and } \frac{r_n(\pi,\lambda,\tau)}{n}\to\frac{\widehat{\mu}_K(K)}{\widehat{\mu}_K(K')}=:\alpha>0. \tag{3.22}$$

Let $K''\subset K'$ be a subset of $(\pi,\lambda,\tau)\in K'$ for which (3.20) and (3.22) hold. Then $\widehat{\mu}_{\mathcal{G}}(K'')=\widehat{\mu}_{\mathcal{G}}(K')>0$. By the ergodicity of $\widehat{\mathcal{R}}$, for a.e. $(\pi,\lambda,\tau)\in X(\mathcal{G})$

$$\text{there exists } n_1(\pi,\lambda,\tau)\geq 0 \text{ such that } \widehat{\mathcal{R}}^{n_1(\pi,\lambda,\tau)}(\pi,\lambda,\tau)\in K''. \tag{3.23}$$

By Fubini argument, there exists a measurable subset $\Xi\subset\mathcal{G}\times\Lambda^{\mathcal{A}}$ such that $\mu_{\mathcal{G}}(\mathcal{G}\times\Lambda^{\mathcal{A}}\setminus\Xi)=0$ and for every $(\pi,\lambda)\in\Xi$ there exists $\tau\in\Theta_\pi$ such that $(\pi,\lambda,\tau)\in X(\mathcal{G})$ satisfies (3.23).

Full measure. We can now show that every $(\pi,\lambda)\in\Xi$ satisfies the UDC. Suppose that $(\pi,\lambda)\in\Xi$ and $(\pi,\lambda,\tau)\in X(\mathcal{G})$ satisfies (3.23). Then the corresponding acceleration sequence $(n_k)_{k\geq 0}$ is defined by setting $n_0:=0$ and then defining n_k inductively such that, for every $k\geq 1$,

$$\widehat{\mathcal{R}}^{n_k}(\pi,\lambda,\tau)=\widehat{\mathcal{R}}_K^{k-1}\widehat{\mathcal{R}}^{n_1(\pi,\lambda,\tau)}(\pi,\lambda,\tau).$$

Let us now consider the cocycle matrices $Z(k)$, $k\in\mathbb{N}$, of the acceleration along the sequence $(n_k)_{k\in\mathbb{N}}$, as defined in § 3.1.3, as well as their products $Q(k,l)$, $k,l\in\mathbb{N}$ (see again § 3.1.3). By definition of Q and (3.7), for $1\leq k\leq l$ we have

$$\begin{aligned}
Q(k,l)&=\widehat{A}_K^{(l-k)}(\widehat{\mathcal{R}}_K^{k-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)))^t\\
Q(0,l)&=\widehat{A}_K^{(l-1)}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))^t\widehat{A}^{(n_1)}(\pi,\lambda,\tau)^t\\
\|Q_s(k,l)\|&=\|\widehat{A}_K^{(l-k)}(\widehat{\mathcal{R}}_K^{k-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)))^t\restriction_{\Gamma_s(\widehat{\mathcal{R}}_K^{k-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)))}\|\\
\|Q_s(0,l)\|&\leq\|\widehat{A}_K^{(l-1)}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))^t\restriction_{\Gamma_s(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))}\|\|A^{(n_1)}(\pi,\lambda)^t\|.
\end{aligned}$$

Since $\widehat{\mathcal{R}}_K^{k-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))\in K$ for every $k\geq 1$, by (3.19), for $0\leq k<l$ we have

$$\|Q_s(k,l)\|\leq Ce^{\lambda}\|A^{(n_1)}(\pi,\lambda)^t\|e^{-\lambda(l-k)},$$

which gives (UDC1).

Consider now the sequence $(r_n)_{n\geq 0}$ defined setting $r_0:=0$ and, for $n\geq 1$,

$$r_n:=r_n(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))+1.$$

As $\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)\in K''$, by (3.22), we have $r_n/n\to\alpha>0$ and

$$\widehat{\mathcal{R}}_K^{r_n-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))=\widehat{\mathcal{R}}_K^{r_n(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))}\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)\in K' \text{ for } n\geq 1.$$

Since $Z(k+1)=\widehat{A}_K(\widehat{\mathcal{R}}_K^{k-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)))^t$ for $k\geq 1$ and $Z(1)=\widehat{A}^{(n_1)}(\pi,\lambda,\tau)^t$, by (3.21), for every $n\geq 1$ and $k\geq 1$ we have

$$\|Z(k+1)\|=\|\widehat{A}_K(\widehat{\mathcal{R}}_K^{k-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)))\|=\|\widehat{A}_K(\widehat{\mathcal{R}}_K^{k-r_n}(\widehat{\mathcal{R}}_K^{r_n-1}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))))\|\leq C'e^{\tau|k-r_n|}.$$

For $k=0$, on the other hand, we have

$$\|Z(1)\|=\|A^{(n_1)}(\pi,\lambda)\|\leq\|A^{(n_1)}(\pi,\lambda)\|e^{\tau|r_n|}$$

for every $n\geq 0$. Moreover, as $r_1=1$, it follows that for every $k\geq 1$ we have

$$\|Z(k+1)\|\leq C'e^{\tau|k-r_1|}\leq C'e^{\tau|k-r_0|},$$

which gives (UDC2) with $C=\max(C',\|A^{(n_1)}(\pi,\lambda)\|)$.

As $\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau)\in K''$, by (3.20)

$$\lim_{k\to+\infty}\frac{\log\|Q(k)\|}{k}=\lim_{k\to+\infty}\frac{\log\|\widehat{A}_K^{(k-1)}(\widehat{\mathcal{R}}^{n_1}(\pi,\lambda,\tau))\|}{k}=\frac{\lambda_1\widehat{\mu}_Y(Y)}{\widehat{\mu}_Y(K)}\in(\lambda_1,\lambda_1(1+\tau)),$$

which implies the condition (UDC3). Finally, the sequence is a Rokhlin-balanced acceleration sequence by Lemma 3.6, since the set Y was chosen to satisfies the conditions (i) and (ii) which guarantee Rokhlin-balance in the proof of Lemma 3.6. This concludes the proof. □

3.3. **Diophantine series.** In the proof of our main results, certain sums and series (defined in Definition 4) which depend on the matrices of the (accelerated) cocycle will play a central role, both to control Birkhoff sums and to prove ergodicity. We here show that these quantities, under the UDC, are first of all well defined and furthermore grow in a controlled way (see Proposition 3.9).

Definition 4. For every IET $T : I \to I$ satisfying Keane's condition and any accelerating sequence we define four sequences $(K_l)_{l\geq -1}$, $(K'_l)_{l\geq -1}$, $(C_k)_{k\geq 0}$, $(C'_k)_{k\geq 0}$:

$$K_l(T) := \sum_{j\geq l} \|Z(j+1)\|\|Q_s(l,j+1)\| \text{ for } l \geq 0 \text{ and } K_{-1} := 0;$$

$$K'_l(T) := \sum_{j\geq l} \|Z(j+1)\|\|Q_s(l,j+1)\| \log\|Q(j)\| \text{ for } l \geq 0 \text{ and } K'_{-1} := 0;$$

$$C_k(T) := \sum_{0\leq l\leq k} \|Q_s(l,k)\|\big(\|Z(l)\|K_{l-1}(T) + K_l(T)\big) \text{ for } k \geq 0;$$

$$C'_k(T) := \sum_{0\leq l\leq k} \|Q_s(l,k)\|\big(\|Z(l)\|K'_{l-1}(T) + K'_l(T)\big) \text{ for } k \geq 0.$$

Proposition 3.9 below shows in particular that if T satisfies the UDC these quantities are finite and hence well defined for every pairs of integers $k \geq 0$, $l \geq -1$.

Proposition 3.9. *For every IET $T : I \to I$ satisfying the UDC all sequences $(K_l)_{l\geq -1}$, $(K'_l)_{l\geq -1}$, $(C_k)_{k\geq 0}$, $(C'_k)_{k\geq 0}$ are well defined and for every $0 < \tau < \lambda/2$ there exists a constant $D > 0$ such that*

$$K_l(T) \leq De^{\tau(r_n - l)} \text{ if } r_{n-1} \leq l \leq r_n \text{ for some } n \geq 0; \tag{3.24}$$

$$K'_l(T) \leq D(l+1)e^{\tau l} \text{ for every } l \geq 0; \tag{3.25}$$

$$C_{r_n}(T) \leq D \text{ for every } n \geq 1; \tag{3.26}$$

$$C'_k(T) \leq D(k+1)e^{2\tau k} \text{ for every } k \geq 0. \tag{3.27}$$

Proof. By (UDC1) and (UDC2), for $r_{n-1} < l \leq r_n$ we have

$$\begin{aligned}
K_l(T) &= \sum_{l+1\leq j\leq r_n} \|Z(j)\|\|Q_s(l,j)\| + \sum_{j>r_n} \|Z(j)\|\|Q_s(l,j)\| \\
&\leq C^2 \sum_{l+1\leq j\leq r_n} e^{\tau(r_n-j+1)}e^{-\lambda(j-l)} + C^2 \sum_{j>r_n} e^{\tau(j-1-r_n)}e^{-\lambda(j-l)} \\
&\leq C^2 e^{\tau(r_n-l)} \sum_{j\geq 1} e^{-\lambda j} + C^2 e^{-\lambda(r_n-l+1)} \sum_{j\geq 0} e^{-(\lambda-\tau)j},
\end{aligned}$$

which gives (3.24).

By condition (UDC3), for all $j \geq l+1$ we have

$$\log\|Q(j)\| \leq \log C + \lambda_1(1+\tau)j \leq C'j \leq C'(l+1)(j-l).$$

Therefore, again by (UDC1) and (UDC2), we have

$$\begin{aligned}
K'_l(T) &\leq C'(l+1) \sum_{j\geq l+1} \|Z(j)\|\|Q_s(l,j)\|(j-l) \\
&\leq C'C^2(l+1) \sum_{j\geq l+1} (j-l)e^{\tau j}e^{-\lambda(j-l)} = C'C^2(l+1)e^{\tau l} \sum_{j\geq 1} je^{-(\lambda-\tau)j},
\end{aligned}$$

which gives (3.25).

In view of (3.24), (UDC1) and (UDC2), we have

$$\begin{aligned}
C_{r_n}(T) &= \sum_{0\leq l\leq r_n} \|Q_s(l,r_n)\|\big(\|Z(l)\|K_{l-1}(T) + K_l(T)\big) \\
&\leq C^2 D \sum_{0\leq l\leq r_n} e^{-\lambda(r_n-l)}\big(e^{\tau(r_n-l+1)}e^{\tau(r_n-l+1)} + e^{\tau(r_n-l)}\big) \leq 2C^2De^{2\tau} \sum_{l\geq 0} e^{-(\lambda-2\tau)l},
\end{aligned}$$

which gives (3.26).

In view of (3.25), (UDC1) and (UDC2), for every $k \geq 0$ we have

$$\begin{aligned}
C'_k(T) &= \sum_{0\leq l\leq k} \|Q_s(l,k)\|\big(\|Z(l)\|K'_{l-1}(T) + K'_l(T)\big) \\
&\leq C^2 D \sum_{0\leq l\leq k} e^{-\lambda(k-l)}\big(le^{\tau l}e^{\tau(l-1)} + (l+1)e^{\tau l}\big) \leq (k+1)2C^2De^{2\tau k} \sum_{j\geq 0} e^{-(\lambda-2\tau)j},
\end{aligned}$$

which gives (3.27). □

4. Cocycles with logarithmic singularities

We define in this section norms on the spaces of cocycles $\varphi : I \to \mathbb{R}$ with logarithmic singularities over IETs that we are interested in (in view of the reduction explained in § 2.3.3). We first introduce (in § 4.1) the class of cocycles of bounded variation over a given IET, then move to cocycles with logarithmic singularities. The norms we introduce make the space of such cocycles a Banach space. We then prove several properties which will be used later in the proofs of the main results.

4.1. **Bounded variation and absolutely continuous cocycles.** Let us denote by $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the space of functions $\varphi : I \to \mathbb{R}$ such that the restriction $\varphi : I_\alpha \to \mathbb{R}$ is of bounded variation for every $\alpha \in \mathcal{A}$.

4.1.1. *Banach structure on bounded variation cocycles.* For every function $\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $x \in I$ we will denote by $\varphi_+(x)$ and $\varphi_-(x)$ the right-handed and left-handed limit of φ at x respectively. Let us denote by $\mathrm{Var}\,(\varphi)|_J$ the total variation of φ on the interval $J \subset I$. Then set

$$\mathrm{Var}\,\varphi := \sum_{\alpha\in\mathcal{A}} \mathrm{Var}\,(\varphi)|_{\mathrm{Int}\, I_\alpha}. \tag{4.1}$$

The space $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is equipped with the Banach norm $\|\varphi\|_{\mathrm{BV}} = \|\varphi\|_{\sup} + \mathrm{Var}\,\varphi$.

4.1.2. *Piecewise absolutely continuous cocycles.* Denote by $\mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the subspace of cocycles in $\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ which are absolutely continuous on the interior of each I_α, $\alpha \in \mathcal{A}$.

Denote by $\mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ the space of functions $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ such that $\varphi' \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. The space $\mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ equipped with the BV norm is a Banach space and $\mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is its dense subspace.

4.1.3. *Boundary operator on cocycles.* Let $\partial_\pi : \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathbb{R}^{\Sigma(\pi)}$ be the linear operator given by

$$(\partial_\pi \varphi)_{\mathcal{O}} := \sum_{\alpha\in\mathcal{A}^-_{\mathcal{O}}} \varphi_-(r_\alpha) - \sum_{\alpha\in\mathcal{A}^+_{\mathcal{O}}} \varphi_+(l_\alpha)$$

for $\mathcal{O} \in \Sigma(\pi)$. This is an extension of the operator defined in § 3.1.9 from piecewise constant cocycles (in view of Remark 3.3) to bounded variation cocycles. It associates to each singularity the *sum of jumps* at the discontinuities associated to that singularity (see also Remark 3.3).

Remark that if $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ then

$$\sum_{\mathcal{O}\in\Sigma(\pi)} (\partial_\pi \varphi)_{\mathcal{O}} = \int_I \varphi'(x)\,dx =: s(\varphi). \tag{4.2}$$

4.2. **Cocycles with logarithmic singularities.** Consider the space $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ of cocycles with logarithmic singularities of geometric type on $\sqcup_{\alpha\in\mathcal{A}} I_\alpha$, defined in § 2.3.1 (see in particular (2.1) for the form of such cocycles), as well as its subspace $\mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, which consist of cocycles with logarithmic singularities of geometric type (see § 2.3.1) satisfying in addition also the symmetry condition (2.2) (both also defined in § 2.3.1). We will also use the spaces

$$\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) := \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) + \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$$
$$\mathrm{LSG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) := \mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) + \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha),$$

consisting of all functions with logarithmic singularities (respectively symmetric logarithmic singularities) of geometric type of the form (2.1) for which we require only that $g_\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. Notice that the space BV (AC resp.) coincides with the subspace of functions $\varphi \in \mathrm{LG}^{\mathrm{BV}}$ (LG resp.) as in (2.1) such that $C^\pm_\alpha = 0$ for all $\alpha \in \mathcal{A}$.

4.2.1. *Norms and Banach space structures.* We now define a norm on $\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ which makes it a Banach space.

Definition 5. For every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ of the form (2.1) set

$$\mathcal{L}(\varphi) := \sum_{\alpha\in\mathcal{A}} (|C^+_\alpha| + |C^-_\alpha|), \qquad \mathcal{LV}(\varphi) := \mathcal{L}(\varphi) + \mathrm{Var}\, g_\varphi.$$

The space $\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ equipped with the norm

$$\|\varphi\|_{\mathcal{LV}} = \mathcal{L}(\varphi) + \|g_\varphi\|_{\mathrm{BV}}$$

becomes a Banach space. Then, since $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ are closed subspaces of $\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, they also inherit the Banach space structure. Moreover, for every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ we have

$$\frac{1}{|I|}\|\varphi\|_{L^1(I)} \le (1 + |\log |I||)\|\|\varphi\|_{\mathcal{LV}}. \tag{4.3}$$

Indeed, since every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is of the form (2.1), we have

$$\frac{1}{|I|}\|\varphi\|_{L^1(I)} \leq \frac{\mathcal{L}(\varphi)}{|I|}\int_I |\log x|\,\mathrm{d}x + \frac{\|g_\varphi\|_{L^1(I)}}{|I|} \leq (1+|\log|I||)\mathcal{L}(\varphi) + \|g_\varphi\|_{\sup}.$$

We can associate a value also to each *saddle* in $\mathrm{Fix}(\psi_\mathbb{R})$ individually as follows. Using the notation introduced in § 3.1.8, let $\mathcal{O} \in \Sigma(\pi)$ be a saddle and let $\mathcal{A}^-_\mathcal{O}, \mathcal{A}^+_\mathcal{O}$ be the sets of letters defined in (3.10), associated respectively to right and left endpoints of intervals which correspond to this saddle. Then

$$\Delta_\mathcal{O}(\varphi) := \sum_{\alpha\in\mathcal{A}^-_\mathcal{O}} C^-_\alpha - \sum_{\alpha\in\mathcal{A}^+_\mathcal{O}} C^+_\alpha, \tag{4.4}$$

is the value of the *asymmetry at the saddle* labelled by $\mathcal{O}$. We also set

$$\mathcal{AS}(\varphi) := \sum_{\mathcal{O}\in\Sigma(\pi)} |\Delta_\mathcal{O}(\varphi)|.$$

Comparing the above definition and (4.4) with Definition 5, one sees that

$$\mathcal{AS}(\varphi) \leq \mathcal{L}(\varphi). \tag{4.5}$$

4.2.2. *Properties of the cocycles arising in the reduction.* As we saw in § 2.3, the study of extensions of locally Hamiltonian flows can be reduced to the study of skew product extensions of IETs with logarithmic singularities (see Proposition 2.4). We now recall the properties of the cocycles which appear from this reduction, which were described in [20] (see the proof of Theorem 6.1 and Proposition 6.1).

Let $M' \subset M$ be a minimal component of a locally Hamiltonian flow $\psi_\mathbb{R}$ with non-degenerate saddles. Fix a section γ as in the proof of Proposition 2.4 and consider the map that associate $f \in C^{2+\epsilon}(M')$ to the cocycle φ_f which appears in the skew-product presentation of the Poincaré map of the extension $\Phi^f_\mathbb{R}$ to $\gamma\times\mathbb{R}$ (see Proposition 2.4).

Proposition 4.1 (Properties of the skew-products cocycles, see [20] and in particular[21] Theorem 6.1)**.** *For every $\epsilon > 0$ the map from $C^{2+\epsilon}(M)$ to $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ which maps*

$$f \mapsto \varphi_f \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$$

is a bounded linear operator. Moreover, $g'_{\varphi_f} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and there exists $C > 0$ such that

$$C^{-1}\sum_{\sigma\in\mathrm{Fix}(\psi_\mathbb{R})\cap M'} |f(\sigma)| \leq \mathcal{L}(\varphi) \leq C\sum_{\sigma\in\mathrm{Fix}(\psi_\mathbb{R})\cap M'} |f(\sigma)| \text{ for every } f \in C^{2+\epsilon}(M).$$

Furthermore:

(i) if $f \in C^1(M)$ and $f(\sigma) = 0$ for all $\sigma \in \mathrm{Fix}(\psi_\mathbb{R}) \cap M'$ then the map $\varphi_{|f|} : I \to \mathbb{R}$ is bounded;

(ii) If $\psi_\mathbb{R} \in \mathcal{U}_{min}$, so $M' = M$, then $\mathcal{AS}(\varphi_f) = 0$ and $\partial_\pi(\varphi_f) = 0$.

4.3. **Properties of cocycles with logarithmic singularities.** We state and prove in this section a number of elementary properties of cocycles with logarithmic singularities which will be used in the construction of the correction operators.

4.3.1. *Control of tails of the derivatives growth.* The derivative of a cocycle with logarithmic singularities has singularities which explode at most as $1/x$, as stated in the following Lemma.

Lemma 4.2. *Suppose that* $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ *and $g_\varphi = 0$. For every $\alpha \in \mathcal{A}$ denote by m_α the middle point of the interval I_α, i.e. $m_\alpha := \frac{1}{2}(l_\alpha + r_\alpha)$. Then*

$$\begin{aligned} |\varphi'(x)(x-l_\alpha)| &\leq \mathcal{L}(\varphi) \text{ for } x \in (l_\alpha, m_\alpha], \\ |\varphi'(x)(x-r_\alpha)| &\leq \mathcal{L}(\varphi) \text{ for } x \in [m_\alpha, r_\alpha). \end{aligned} \tag{4.6}$$

Proof. Indeed, for every $x \in (l_\alpha, m_\alpha]$ and $\beta \in \mathcal{A}$ we have

$$\left\{\frac{x-l_\beta}{|I|}\right\} \geq \frac{x-l_\alpha}{|I|},\ \left\{\frac{r_\beta - x}{|I|}\right\} \geq \frac{r_\alpha - x}{|I|} \geq \frac{x-l_\alpha}{|I|}.$$

It follows that

$$|\varphi'(x)(x-l_\alpha)| \leq \sum_{\beta\in\mathcal{A}} \frac{|C^+_\beta|(x-l_\alpha)}{|I|\{(x-l_\beta)/|I|\}} + \sum_{\beta\in\mathcal{A}} \frac{|C^-_\beta|(x-l_\alpha)}{|I|\{(r_\beta - x)/|I|\}} \leq \sum_{\beta\in\mathcal{A}} (|C^+_\beta| + |C^-_\beta|) = \mathcal{L}(\varphi).$$

The second inequality of (4.6) follows by the same arguments. □

[21] The statements are all part of Theorem 6.1 in [20], but (*i*), namely the boundedness of $\varphi_{|f|}$ when $f \in C^1(M)$ and f vanishes on $\mathrm{Fix}(\psi_\mathbb{R}) \cap M'$. This last statement can be proved with the same arguments used in [20] to prove Theorem 6.1.

4.3.2. *Control of mean value on subintervals.* For every integrable function $f : I \to \mathbb{R}$ and a subinterval $J \subset I$ let $m(f, J)$ stand for the *mean value* of f on J, i.e.

$$m(f,J) = \frac{1}{|J|}\int_J f(x)\,dx.$$

Proposition 4.3 (Proposition 2.5 in [20]). *If $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $J \subset I_\alpha$ for some $\alpha \in \mathcal{A}$, then*

$$|m(\varphi,J) - m(\varphi,I_\alpha)| \le \mathcal{LV}(\varphi)\left(4 + \frac{|I_\alpha|}{|J|}\right) \tag{4.7}$$

and

$$\frac{1}{|J|}\int_J |\varphi(x) - m(\varphi,J)|\,dx \le 8\mathcal{LV}(\varphi). \tag{4.8}$$

Lemma 4.4. *Let $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. Then for every $x \in \operatorname{Int} I_\alpha$ we have*

$$\begin{aligned} |\varphi(x) - m(\varphi,[l_\alpha,m_\alpha])| &\le \mathcal{LV}(\varphi)\Big(1 + \log\frac{|I_\alpha|}{(x-l_\alpha)}\Big) \text{ if } x \in (l_\alpha, m_\alpha],\\ |\varphi(x) - m(\varphi,[m_\alpha,r_\alpha])| &\le \mathcal{LV}(\varphi)\Big(1 + \log\frac{|I_\alpha|}{(r_\alpha-x)}\Big) \text{ if } x \in [m_\alpha, r_\alpha). \end{aligned} \tag{4.9}$$

Proof. Step 1: First note that for any C^1-map $f : (x_0, x_1] \to \mathbb{R}$ such that $|f'(x)(x - x_0)| \le C$ for $x \in (x_0, x_1]$, we have that for all $t, s \in (x_0, x_1]$

$$|f(s) - f(t)| = |\int_t^s f'(u)\,du| \le C|\int_t^s \frac{1}{u-x_0}\,du| = C|\log\frac{t-x_0}{s-x_0}|$$

and hence that

$$\begin{aligned} |f(s) - m(f,[x_0,x_1])| &\le \frac{C}{x_1-x_0}\int_{x_0}^{x_1} |\log\frac{t-x_0}{s-x_0}|\,dt\\ &= C\Big(\log\frac{x_1-x_0}{s-x_0} + 1 - 2\frac{x_1-s}{x_1-x_0}\Big) \le C\Big(\log\frac{x_1-x_0}{s-x_0} + 1\Big). \end{aligned} \tag{4.10}$$

Step 2: Suppose now that $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $g_\varphi = 0$. In view of Lemma 4.2 (see (4.6)), we can apply (4.10) to $f = \varphi$ restricted to I_α and taking $C = \mathcal{LV}(\varphi) = \mathcal{L}(\varphi)$. This gives (4.9) in the case $g_\varphi = 0$.
Step 3: Consider now the general case. For every $g \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and any interval $J \subset I_\alpha$, we have

$$|g(x) - m(g,J)| \le \mathrm{Var}(g) \text{ for every } x \in J. \tag{4.11}$$

Adding this equality to the result of *Step 2*, we obtain (4.9) for any $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. □

From Lemma 4.4 and (4.11), we immediately get the following Corollary:

Corollary 4.5. *Let $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. Then for every $x \in \operatorname{Int} I_\alpha$ we have*

$$|\varphi(x)| \le \frac{2|I|}{|I_\alpha|}\frac{\|\varphi\|_{L^1(I)}}{|I|} + \mathcal{LV}(\varphi)\Big(1 + \log\frac{|I_\alpha|}{\min\{x-l_\alpha, r_\alpha - x\}}\Big). \tag{4.12}$$

If additionally $\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ then

$$\|\varphi\|_{\sup} \le \frac{|I|}{\min_{\alpha\in\mathcal{A}}|I_\alpha|}\frac{\|\varphi\|_{L^1(I)}}{|I|} + \mathrm{Var}(\varphi). \tag{4.13}$$

4.3.3. *Extenstion of the boundary opeartor.* The operator $\partial_\pi : \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathbb{R}^{\Sigma(\pi)}$ introduced in § 4.1.3 can be extended to an operator $\partial_\pi : \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathbb{R}^{\Sigma(\pi)}$ as follows.

Definition 6. Let $\partial_\pi : \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathbb{R}^{\Sigma(\pi)}$ be a linear operator given by

$$\partial_\pi(\varphi)_{\mathcal{O}} := \lim_{x\to 0^+}\Big(\sum_{\alpha\in\mathcal{A}^-_{\mathcal{O}}}\big(\varphi(r_\alpha - x) + C^-_\alpha\log x\big) - \sum_{\alpha\in\mathcal{A}^+_{\mathcal{O}}}\big(\varphi(l_\alpha + x) + C^+_\alpha\log x\big)\Big).$$

for every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\mathcal{O} \in \Sigma(\pi)$.

Let $a := \min\{|I_\beta| : \beta \in \mathcal{A}\}/2$. Then for every $\alpha \in \mathcal{A}$ and every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ there are $\underline{\varphi}^+_\alpha, \underline{\varphi}^-_\alpha :$ $[0, a] \to \mathbb{R}$ functions of bounded variation such that

$$\varphi(r_\alpha - x) = -C^-_\alpha\log x + \underline{\varphi}^-_\alpha(x),\ \ \varphi(l_\alpha + x) = -C^+_\alpha\log x + \underline{\varphi}^+_\alpha(x) \text{ for } x \in (0, a].$$

For every $\mathcal{O} \in \Sigma(\pi)$ let us consider the bounded variation map $D_{\mathcal{O}} : [0, a] \to \mathbb{R}$ given by

$$D_{\mathcal{O}}(x) := \sum_{\alpha\in\mathcal{A}^-_{\mathcal{O}}}\underline{\varphi}^-_\alpha(x) - \sum_{\alpha\in\mathcal{A}^+_{\mathcal{O}}}\underline{\varphi}^+_\alpha(x) \text{ for } x \in [0, a]. \tag{4.14}$$

Then for all $x \in (0, a]$ we have

$$D_{\mathcal{O}}(x) = \sum_{\alpha\in\mathcal{A}^-_{\mathcal{O}}} \big(\varphi(r_\alpha - x) + C^-_\alpha \log x\big) - \sum_{\alpha\in\mathcal{A}^+_{\mathcal{O}}} \big(\varphi(l_\alpha + x) + C^+_\alpha \log x\big). \tag{4.15}$$

As $D_{\mathcal{O}}$ is of bounded variation, it follows that

$$\partial_\pi(\varphi)_{\mathcal{O}} = (D_{\mathcal{O}})_+(0) \tag{4.16}$$

is well defined.

4.4. **Mean value projection.** If $\varphi \in L^1(I)$, we can consider the piecewise constant function that is constant and equal to the mean $m(\varphi, I_\alpha)$ on I_α. Formally, we define the linear operator $\mathcal{M} : L^1(I) \to \mathbb{R}^{\mathcal{A}}$ given by

$$\mathcal{M}(\varphi)(x) = m(\varphi, I_\alpha) \text{ if } x \in I_\alpha.$$

This operator will play an important role in defining *corrections* operators. In the rest of this subsection we prove the following Proposition, that gives an estimate on how the boundary operator ∂_π changes when one projects using this mean value projection operator $\mathcal{M}$.

Proposition 4.6. *For every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ we have*

$$\|\partial_\pi(\mathcal{M}\varphi)\| \le \|\partial_\pi(\varphi)\| + \mathcal{AS}(\varphi)\Big(1 + \log\frac{2}{\min_{\beta\in\mathcal{A}}|I_\beta|}\Big) + 2d\mathcal{LV}(\varphi)\left(5 + 2\frac{|I|}{\min_{\beta\in\mathcal{A}}|I_\beta|}\right). \tag{4.17}$$

Furthermore, we also have that

$$\|\partial_\pi(\varphi)\| \le 2d \log\frac{2}{\min_{\beta\in\mathcal{A}}|I_\beta|}\|\varphi\|_{\mathcal{LV}}. \tag{4.18}$$

Proof. First suppose that $g_\varphi = 0$. Then the maps $\underline{\varphi}^\pm_\alpha : [0, a] \to \mathbb{R}$ ($a := \min\{|I_\beta| : \beta \in \mathcal{A}\}/2$) are of class C^1 for all $\alpha \in \mathcal{A}$ with

$$|\underline{\varphi}^\pm_\alpha(x)| \le \mathcal{L}(\varphi)\log a^{-1} \quad \text{and} \quad |(\underline{\varphi}^\pm_\alpha)'(x)| \le \mathcal{L}(\varphi)/a \quad \text{for} \quad x \in [0, a]. \tag{4.19}$$

In view of (4.14) and (4.16), it follows that for every $\mathcal{O} \in \Sigma(\pi)$ the map $D_{\mathcal{O}}$ is of class C^1 and we have

$$|\partial_\pi(\varphi)_{\mathcal{O}}| = |D_{\mathcal{O}}(0)| \le (\#\mathcal{A}^+_{\mathcal{O}} + \#\mathcal{A}^-_{\mathcal{O}})\mathcal{L}(\varphi)\log a^{-1} \tag{4.20}$$

and

$$|D'_{\mathcal{O}}(x)| \le \frac{(\#\mathcal{A}^+_{\mathcal{O}} + \#\mathcal{A}^-_{\mathcal{O}})\mathcal{L}(\varphi)}{a} \text{ for } x \in [0, a].$$

Therefore, for every $x \in [0, a]$,

$$|D_{\mathcal{O}}(0) - m(D_{\mathcal{O}}, [0, a])| \le \frac{\int_0^a |D_{\mathcal{O}}(0) - D_{\mathcal{O}}(x)|\,dx}{a} \le \frac{\int_0^a\int_0^x |D'_{\mathcal{O}}(s)|\,ds\,dx}{a} \le (\#\mathcal{A}^+_{\mathcal{O}} + \#\mathcal{A}^-_{\mathcal{O}})\mathcal{L}(\varphi). \tag{4.21}$$

Moreover, by (4.15) and (4.4), we have

$$m(D_{\mathcal{O}}, [0, a]) = \Delta_{\mathcal{O}}(\varphi)m(\log, [0, a]) + \sum_{\alpha\in\mathcal{A},\pi_0(\alpha)\in\mathcal{O}} m(\varphi, [r_\alpha - a, r_\alpha]) - \sum_{\alpha\in\mathcal{A},\pi_0(\alpha)-1\in\mathcal{O}} m(\varphi, [l_\alpha, l_\alpha + a]).$$

In view of (4.7), for every $\alpha \in \mathcal{A}$ we have

$$|m(\varphi, [r_\alpha - a, r_\alpha]) - m(\varphi, I_\alpha)| \le \mathcal{L}(\varphi)\Big(4 + \frac{|I_\alpha|}{a}\Big)$$
$$|m(\varphi, [l_\alpha, l_\alpha + a]) - m(\varphi, I_\alpha)| \le \mathcal{L}(\varphi)\Big(4 + \frac{|I_\alpha|}{a}\Big).$$

As $m(\log, [0, a]) = \log a - 1$, it follows that

$$|\partial_\pi(\mathcal{M}\varphi)_{\mathcal{O}} - m(D_{\mathcal{O}}, [0, a])| \le |\Delta_{\mathcal{O}}(\varphi)|(1 + \log a^{-1}) + \mathcal{L}(\varphi)\Big(4(\#\mathcal{A}^+_{\mathcal{O}} + \#\mathcal{A}^-_{\mathcal{O}}) + 2\frac{|I|}{a}\Big).$$

Together with (4.16) and (4.21), this gives

$$|\partial_\pi(\mathcal{M}\varphi)_{\mathcal{O}} - \partial_\pi(\varphi)_{\mathcal{O}}| \le |\Delta_{\mathcal{O}}(\varphi)|(1 + \log a^{-1}) + \mathcal{L}(\varphi)\Big(5(\#\mathcal{A}^+_{\mathcal{O}} + \#\mathcal{A}^-_{\mathcal{O}}) + 2\frac{|I|}{a}\Big).$$

As $\sum_{\mathcal{O}\in\Sigma(\pi)}(\#\mathcal{A}^+_{\mathcal{O}} + \#\mathcal{A}^-_{\mathcal{O}}) = 2\#\mathcal{A} = 2d$ and $\mathcal{AS}(\varphi) = \sum_{\mathcal{O}\in\Sigma(\pi)}|\Delta_{\mathcal{O}}(\varphi)|$, summing up these inequalities for all $\mathcal{O} \in \Sigma(\pi)$ we have

$$\|\partial_\pi(\mathcal{M}\varphi) - \partial_\pi(\varphi)\| \le \mathcal{AS}(\varphi)(1 + \log a^{-1}) + 2d\mathcal{L}(\varphi)\left(5 + \frac{|I|}{a}\right). \tag{4.22}$$

Now assume that $g_\varphi \neq 0$. Since $g_\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, we have

$$|(g_\varphi)_+(l_\alpha) - m(g_\varphi, I_\alpha)| \le \operatorname{Var} g_\varphi \quad \text{and} \quad |(g_\varphi)_-(r_\alpha) - m(g_\varphi, I_\alpha)| \le \operatorname{Var} g_\varphi.$$

It follows that for every $\mathcal{O}\in\Sigma(\pi)$ we have

$$\begin{aligned}\big|\partial_\pi(\mathcal{M}(g_\varphi))_{\mathcal{O}}-\partial_\pi(g_\varphi)_{\mathcal{O}}\big| &= \Big|\sum_{\pi_0(\alpha)\in\mathcal{O}}\big(m(g_\varphi,I_\alpha)-(g_\varphi)_-(r_\alpha)\big)-\sum_{\pi_0(\alpha)-1\in\mathcal{O}}\big(m(g_\varphi,I_\alpha)-(g_\varphi)_+(l_\alpha)\big)\Big|\\ &\leq (\#\mathcal{A}^+_{\mathcal{O}}+\#\mathcal{A}^-_{\mathcal{O}})\operatorname{Var} g_\varphi.\end{aligned}$$

Summing up these inequalities for all $\mathcal{O}\in\Sigma(\pi)$ we have

$$\|\partial_\pi(\mathcal{M}(g_\varphi))-\partial_\pi(g_\varphi)\|\leq 2d\operatorname{Var} g_\varphi \tag{4.23}$$

which together with (4.22) this completes the proof of (4.17).

By the definition of $\partial_\pi(g_\varphi)_{\mathcal{O}}$, we also have

$$|\partial_\pi(g_\varphi)_{\mathcal{O}}|\leq(\#\mathcal{A}^+_{\mathcal{O}}+\#\mathcal{A}^-_{\mathcal{O}})\|g_\varphi\|_{\sup}\quad\text{for every}\quad\mathcal{O}\in\Sigma(\pi).$$

This, together with (4.20), gives

$$\|\partial_\pi(\varphi)\|\leq 2d\Big(\mathcal{L}(\varphi)\frac{2}{\min_{\beta\in\mathcal{A}}|I_\beta|}+\|g_\varphi\|_{\sup}\Big).$$

As $\|\varphi\|_{\mathcal{LV}}=\mathcal{L}(\varphi)+\operatorname{Var} g_\varphi+\|g_\varphi\|_{\sup}$, this completes the proof of (4.18). □

5. Renormalization of cocycles

The renormalization map on IETs given by Rauzy-Veech induction (or any of its accelerations) induce also a renormalization operator on cocycles over IETs defined in § 3.1.5.

5.1. Special Birkhoff sums.

Recall that for all $0\leq k<l$ the renormalization operator $S(k,l):L^1(I^{(k)})\to L^1(I^{(l)})$ is given by

$$S(k,l)\varphi(x)=\sum_{0\leq i<Q_\beta(k,l)}\varphi((T^{(k)})^ix)\text{ for }x\in I^{(l)}_\beta.$$

We write $S(k)\varphi$ for $S(0,k)\varphi$ and we use the convention that $S(k,k)\varphi:=\varphi$. Sums of this form are usually called *special Birkhoff sums*. Since Rokhlin towers representation allows to write $I^{(k)}$ as

$$I^{(k)}=\bigcup_{\beta\in\mathcal{A}}\bigcup_{i=0}^{Q_\beta(k,l)-1}(T^{(k)})^iI^{(l)}_\beta,$$

where the intervals in the union are all pairwise disjoint, from the definition of special Birkhoff sums, one can see that for every $\varphi\in L^1(I^{(k)})$ we have

$$\int_{I^{(l)}}S(k,l)\varphi(x)\,dx=\int_{I^{(k)}}\varphi(x)\,dx. \tag{5.1}$$

Therefore we also have that

$$\|S(k,l)\varphi\|_{L^1(I^{(l)})}\leq\|\varphi\|_{L^1(I^{(k)})}. \tag{5.2}$$

If $g\in\mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}}I^{(k)}_\alpha)$ then

$$\operatorname{Var} S(k,l)g\leq\operatorname{Var} g. \tag{5.3}$$

The following Lemma, which was proved by the authors in [20], shows that *constants* of logarithmic singularities, as a *set*, is invariant under renormalization when logarithmic singularities are normalized suitably (i.e. by the map $f(x)\mapsto f(\lambda\{x/\lambda\})$, where λ is the length of the inducing interval).

Lemma 5.1 (see [20])**.** *For each $0\leq k\leq l$ and for each $\varphi\in\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I^{(k)}_\alpha)$ of the form*

$$\varphi(x)=-\sum_{\alpha\in\mathcal{A}}\left(C^+_\alpha\log\left(|I^{(k)}|\left\{\frac{x-l^{(k)}_\alpha}{|I^{(k)}|}\right\}\right)+C^-_\alpha\log\left(|I^{(k)}|\left\{\frac{r^{(k)}_\alpha-x}{|I^{(k)}|}\right\}\right)\right) \tag{5.4}$$

there exists a permutation $\chi:\mathcal{A}\to\mathcal{A}$ such that

$$S(k,l)\varphi(x)=-\sum_{\alpha\in\mathcal{A}}C^+_\alpha\log\big(|I^{(l)}|\{(x-l^{(l)}_\alpha)/|I^{(l)}|\}\big)-\sum_{\alpha\in\mathcal{A}}C^-_{\chi(\alpha)}\log\big(|I^{(l)}|\{(r^{(l)}_\alpha-x)/|I^{(l)}|\}\big)+g_{S(k,l)\varphi}(x),$$

where $g_{S(k,l)\varphi}\in\mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}}I^{(l)}_\alpha)$.

Remark 5.2. In the general case, when $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and g_φ is non-trivial, the map $\varphi - g_\varphi$ is of the form (5.4). It follows that

$$\begin{aligned} S(k,l)\varphi(x) =& S(k,l)(\varphi - g_\varphi)(x) + S(k,l)(g_\varphi)(x) \\ =& -\sum_{\alpha\in\mathcal{A}} C_\alpha^+ \log\left(|I^{(l)}|\{(x - l_\alpha^{(l)})/|I^{(l)}|\}\right) - \sum_{\alpha\in\mathcal{A}} C_{\chi(\alpha)}^- \log\left(|I^{(l)}|\{(r_\alpha^{(l)} - x)/|I^{(l)}|\}\right) \\ &+ g_{S(k,l)(\varphi-g_\varphi)}(x) + S(k,l)(g_\varphi)(x). \end{aligned}$$

As $g_{S(k,l)(\varphi-g_\varphi)}$ and $S(k,l)(g_\varphi)$ belong to $\mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$, we have

$$g_{S(k,l)\varphi} = g_{S(k,l)(\varphi-g_\varphi)} + S(k,l)(g_\varphi). \tag{5.5}$$

Recalling the definition of $\mathcal{L}$ and $\mathcal{AS}$ (see Definition 5) and of the various spaces of cocycles with logarithmic singularities (refer to § 4), we immediately have the following corollary:

Corollary 5.3 (Invariance of $\mathcal{L}$ and $\mathcal{AS}$). *For every* $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$

$$\mathcal{L}(S(k,l)\varphi) = \mathcal{L}(\varphi) \quad \text{and} \quad \mathcal{AS}(S(k,l)\varphi) = \mathcal{AS}(\varphi). \tag{5.6}$$

Therefore, the operator $S(k,l)$ *maps:*

(i) the space $\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ *into the space* $\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$*;*
(ii) the space $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ *into the space* $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$*;*
(iii) the space $\mathrm{LSG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ *into the space* $\mathrm{LSG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$*;*
(iv) the space $\mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ *into the space* $\mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(l)})$*.*

The following result (Lemma 5.4) is a generalization of Lemma 3.2 in [20], which was proved for cocycles with strongly symmetric logarithmic singularities. Since the proof of the following lemma runs in the same way, we skip it. The operator ∂_π which appears in the statement was defined in § 4.3.3.

Lemma 5.4. *For all* $0 \le k \le l$ *and for every* $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ *we have*

$$\|\partial_{\pi^{(l)}}(S(k,l)\varphi)\| = \|\partial_{\pi^{(k)}}(\varphi)\|. \tag{5.7}$$

5.2. **Cancellations for symmetric singularities.** The following property of cocycles with symmetric logarithmic singularities was proved by the second author in [63] (see Proposition 4.1) and will play a crucial role to renormalize cocycles with symmetric logarithmic singularities and in the proof of ergodicity.

Let us denote by $(x)^+$ the positive part of x, i.e. $(x)^+ = x$ if $x \ge 0$ and $(x)^+ = \infty$ if $x < 0$, so that if $x < 0$ then $1/(x)^+$ is zero. Using this notation, let us define, for every $\alpha \in \mathcal{A}$,

$$x_\alpha^l := \min_{0\le i<Q_\beta(k)} (T^i x - l_\alpha)^+, \qquad x_\alpha^r := \min_{0\le i<Q_\beta(k)} (r_\alpha - T^i x)^+. \tag{5.8}$$

Then x_α^l (resp. x_α^r) is the *closest visit* to the singularity l_α from the right (resp. to r_α from the left) in the orbit segment $\{T^i(x),\ 0 \le i < Q_\beta(k)\}$.

Remark 5.5 (Closest visits comparison). By the proof of Proposition 3.2 in [20], for every $x \in I_\beta^{(k)}$ and any $\alpha \in \mathcal{A}$ we have

$$\left|\frac{1}{x_\alpha^l} - \frac{1}{\left|I^{(k)}\right|\left\{\frac{x-l_\alpha^{(k)}}{|I^{(k)}|}\right\}}\right| \le \frac{1}{\left|I_\alpha^{(k)}\right|}, \quad \left|\frac{1}{x_{\chi(\alpha)}^r} - \frac{1}{\left|I^{(k)}\right|\left\{\frac{r_\alpha^{(k)}-x}{|I^{(k)}|}\right\}}\right| \le \frac{1}{\left|I_\alpha^{(k)}\right|}. \tag{5.9}$$

Thus, the closests visits defined above are comparable with the quantities expressed above in terms of $\{\cdot\}$.

The following Theorem (as the proof below indicates) follows from the results in [63], combined with the acceleration defined in the UDC:

Theorem 5.6 (Cancellations for Symmetric Logarithmic Singularities). *For almost every* $(\pi,\lambda) \in \mathcal{G} \times \mathbb{R}_{>0}^{\mathcal{A}}$ *there exists an accelerating sequence and a constant* $M = M_{(\pi,\lambda)} \ge 1$ *such that* $T_{(\pi,\lambda)}$ *satisfies the UDC (along the accelerating sequence) and for every* $\varphi \in \mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ *with* $g'_\varphi = 0$*, any* $k \ge 1$ *and* $x \in I_\beta^{(k)}$ *we have*

$$\left|(\varphi')^{(Q_\beta(k))}(x) - \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^+}{x_\alpha^l} + \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^-}{x_\alpha^r}\right| \le M\mathcal{L}(\varphi)\frac{Q_\beta(k)}{|I|}, \tag{SUDC1}$$

where x_α^l *and* x_α^r *are the closets visits defined in* (5.8)*.*

Moreover, for every $0 \le r < Q_\beta(k)$ *and* $x \in I_\beta^{(k)}$ *we have*

$$\left|(\varphi')^{(r)}(x)\right| \le \sum_{\alpha\in\mathcal{A}} \frac{|C_\alpha^+|}{x_\alpha^l} + \sum_{\alpha\in\mathcal{A}} \frac{|C_\alpha^-|}{x_\alpha^r} + M\mathcal{L}(\varphi)\frac{Q_\beta(k)}{|I|}. \tag{SUDC2}$$

Proof. By the proof of Propositions 4.1 and 4.2 in [63], there exists a precompact subset $E_D \subset X(\mathcal{G})$ with positive measure such that $\widehat{A}_{E_D}$ and $\widehat{A}_{E_D}^{-1}$ are log-integrable and the accelerating sequence defined by recurrence of (π, λ, τ) to E_D is such that (SUDC1) and (SUDC2) hold for every $k \geq 1$.

Then we repeat all steps of the proof of Theorem 3.8 starting from the set $Y = E_D$. Since both (SUDC1) and (SUDC2) also holds along a subsequence obtained taking further accelerations, this completes the proof. □

Definition 7 (SUDC). We say that an IET T satisfies the *Symmetric Uniform Diophantine Condition*, or SUDC for short, if it satisfies the UDC along an accelerating sequence $(n_k)_{k\geq 0}$ along which the cancellations (SUDC1) and (SUDC2) hold.

Theorem 5.6 above thus shows that the SUDC has full measure.

Proposition 5.7. *Suppose that T satisfies the SUDC. For every $\varphi \in \mathrm{LSG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $g_\varphi = 0$ and $k \geq 1$ we have $g_{S(k)\varphi} \in \mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and*

$$\|g'_{S(k)\varphi}\|_{\sup} \leq \frac{(M+1)\mathcal{L}(\varphi)}{\min_{\beta\in\mathcal{A}} |I_\beta^{(k)}|}. \tag{5.10}$$

Proof. The proof runs in the same way as the proof of Proposition 3.2 in [20], only replacing Corollary 3.1 in [20] with (SUDC1).

Let $\chi : \mathcal{A} \to \mathcal{A}$ be the permutation given by Lemma 5.1. Then

$$g'_{S(k)\varphi}(x) = S(k)\varphi'(x) - \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^+}{|I^{(k)}|\left\{\frac{x-l_\alpha^{(k)}}{|I^{(k)}|}\right\}} + \sum_{\alpha\in\mathcal{A}} \frac{C_{\chi(\alpha)}^-}{|I^{(k)}|\left\{\frac{r_\alpha^{(k)}-x}{|I^{(k)}|}\right\}}. \tag{5.11}$$

Notice that $S(k)\varphi'(x) = (\varphi')^{(Q_\beta(k))}(x)$ if $x \in I_\beta^{(k)}$. Thus, (5.11), in view of (SUDC1) and Remark 5.5, and remarking that (since Rokhlin towers give a partition)

$$Q_\beta(k) \min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}| \leq Q_\beta(k)|I_\beta^{(k)}| \leq \sum_{\alpha\in\mathcal{A}} Q_\alpha(k)|I_\alpha^{(k)}| = |I|,$$

we get that, for every $x \in I_\beta^{(k)}$

$$|g'_{S(k)\varphi}(x)| \leq M\mathcal{L}(\varphi)\frac{Q_\beta(k)}{|I|} + \frac{\mathcal{L}(\varphi)}{\min_\alpha |I_\alpha^{(k)}|} \leq (M+1)\frac{\mathcal{L}(\varphi)}{\min_\alpha |I_\alpha^{(k)}|}.$$

Taking the supremum over $x \in I^{(k)}$ concludes the proof. □

Proposition 5.8. *If T satisfies the SUDC then for every $k \geq 1$ and for every $\varphi \in \mathrm{LSG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ we have*

$$\mathcal{LV}(S(k)\varphi) \leq 4M\frac{|I^{(k)}|}{\min_{\beta\in\mathcal{A}} |I_\beta^{(k)}|}\mathcal{LV}(\varphi) \leq 4M\kappa\mathcal{LV}(\varphi). \tag{5.12}$$

Proof. First suppose that $g_\varphi = 0$. By Proposition 5.7, we then have that $g_{S(k)\varphi}$ belongs to the space $\mathrm{BV}^1(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and

$$\operatorname{Var} g_{S(k)\varphi} = \int_{I^{(k)}} |g'_{S(k)\varphi}(x)|\, dx \ \leq \|g'_{S(k)\varphi}\|_{\sup}|I^{(k)}| \leq (M+1)\mathcal{L}(\varphi)\frac{|I^{(k)}|}{\min_{\beta\in\mathcal{A}} |I_\beta^{(k)}|}.$$

If $g_\varphi \neq 0$ then, by (5.3), we have $\operatorname{Var}(S(k)g_\varphi) \leq \operatorname{Var} g_\varphi$. As $g_{\varphi - g_\varphi} = 0$, by (5.5), it follows that

$$\begin{aligned}\mathcal{LV}(S(k)\varphi) &= \mathcal{L}(S(k)\varphi) + \operatorname{Var} g_{S(k)\varphi} = \mathcal{L}(\varphi) + \operatorname{Var}(g_{S(k)(\varphi-g_\varphi)} + S(k)g_\varphi)\\ &\leq \mathcal{L}(\varphi) + \operatorname{Var}(g_{S(k)(\varphi-g_\varphi)}) + \operatorname{Var}(S(k)g_\varphi).\end{aligned}$$

Therefore, by Proposition 5.7, we get

$$\mathcal{LV}(S(k)\varphi) \leq \mathcal{L}(\varphi) + (M+1)\frac{|I^{(k)}|}{\min_{\beta\in\mathcal{A}} |I_\beta^{(k)}|}\mathcal{L}(\varphi - g_\varphi) + \operatorname{Var}(g_\varphi) \leq 4M\frac{|I^{(k)}|}{\min_{\beta\in\mathcal{A}} |I_\beta^{(k)}|}\mathcal{LV}(\varphi).$$

□

5.3. **Non-symmetric case.** We now estimate Birkhoff sums for the derivative φ' of a function $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ with *asymmetric* logarithmic singularities. Birkhoff sums of this type of function over *rotations* (which can be thought as IETs with $d = 2$) were first estimated in the seminar work by Kocergin [37] (see also [38]). When the base transformation is an IET, they were studied by the second author in [61] when there is a unique logarithmic singularity and by Ravotti in [56] in the general case. A crucial estimate in all these works is provided by the following Remark, which was first used by Kocergin in [37].

Remark 5.9 (Inverses of an arithmetic progression). If the points $(x_i)_{i=0}^N \subset [0,1]$ are such that, for some $\delta > 0$, $|x_i - x_j| \geq \delta$ for every pair of $i \neq j$, then

$$\sum_{i=0}^{N} \frac{1}{x_i} \leq \frac{1}{\min_{0\leq i\leq N} x_i} + \sum_{j=1}^{N} \frac{1}{j\,\delta} \leq \frac{1}{\min_{0\leq i\leq N} x_i} + \frac{\log N + 1}{\delta}.$$

Lemma 5.10. *Suppose that $T_{(\pi,\lambda)}$ satisfies the Keane condition. Then for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $g'_\varphi = 0$, any $k \geq 1$ and $x \in I^{(k)}$ we have*

$$\left| S(k)(\varphi')(x) - \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^+}{x_\alpha^l} + \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^-}{x_\alpha^r} \right| \leq \mathcal{L}(\varphi) \frac{1 + \log \|Q(k)\|}{\min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|}. \tag{5.13}$$

Proof. Notice first that it is enough to prove (5.13) in the special cases when

$$\varphi = \varphi_\alpha^+ := \log(|I|\{(x - l_\alpha)/|I|\}) \text{ or } \varphi = \varphi_\alpha^- := \log(|I|\{(r_\alpha - x)/|I|\}).$$

Indeed, taking the linear combination $\sum_{\alpha\in\mathcal{A}} C_\alpha^+ \varphi_\alpha^+ + C_\alpha^- \varphi_\alpha^-$ then yields the general form of the result. Since the reasoning is analogous for functions of the form φ_α^+ or φ_α^- we will only do the computations for φ_α^+.

For any $x \in I_\beta^{(k)}$ choose $0 \leq i_0 < Q_\beta(k)$ such that the iterate $T^{i_0}x$ is the closest to l_α among all iterates $T^j x$ with $0 \leq j < Q_\beta(k)$ belonging to the interval $(l_\alpha, |I|)$. Then $x_\alpha^l = T^{i_0}(x) - l_\alpha$. Since all points in the orbit segment $\{T^k x,\ 0 \leq k < Q_\beta(k)\}$ belong to separate floors of a Rokhlin tower on which T acts as an isometry on the floors, we also have that

$$\min\{|T^i(x) - T^j(x)|, 0 \leq i \neq j < Q_\beta(k)\} \geq \min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|.$$

Therefore, if we reorder the points in $\{T^i x\, 0 \leq i < Q_\beta(k)\}$ so that $l_\alpha < T^{i_0}x < T^{i_1}x < T^{i_2}x < \ldots$, we have

$$|I|\{(T^{i_j}(x) - l_\alpha)/|I|\} \geq \min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|\, j \qquad \text{for all } 1 \leq j < Q_\beta(k).$$

Thus, since by definition of special Birkhoff sum $S(k)\varphi'(x) = (\varphi')^{(Q_\beta(k))}(x)$ if $x \in I_\beta^{(k)}$,

$$\left|(\varphi')^{(Q_\beta(k))}(x) - \frac{1}{x_\alpha^l}\right| \leq \sum_{0\leq i<Q_\beta(k), i\neq i_0} \frac{1}{|I|\{(T^i x - l_\alpha)/|I|\}} \leq \sum_{1\leq j<Q_\beta(k)} \frac{1}{j\, \min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|} \leq \frac{1 + \log Q_\beta(k)}{\min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|},$$

were in the last inequality we have used the estimate given by Remark 5.9. This completes the proof. □

Lemma 5.11. *Suppose that $T_{(\pi,\lambda)}$ satisfies the Keane condition. Then for every $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $k \geq 1$ we have*

$$\mathcal{LV}(S(k)\varphi) \leq \frac{|I^{(k)}|}{\min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|} \mathcal{LV}(\varphi)(3 + \log \|Q(k)\|). \tag{5.14}$$

Proof. First suppose that $g_\varphi = 0$. Then, by Lemma 5.1 and in view of Remark 5.5, we can apply the derivative estimates given by Lemma 5.10 and get that for every $x \in I^{(k)}$,

$$\begin{aligned}|g'_{S(k)\varphi}(x)| &= \left| S(k)\varphi'(x) - \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^+}{|I^{(k)}|\{(x - l_\alpha^{(k)})/|I^{(k)}|\}} + \sum_{\alpha\in\mathcal{A}} \frac{C_{\chi(\alpha)}^-}{|I^{(k)}|\{(r_\alpha^{(k)} - x)/|I^{(k)}|\}} \right| \\ &\leq \frac{\mathcal{L}(\varphi)(2 + \log \|Q(k)\|)}{\min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|}.\end{aligned}$$

It follows that

$$\operatorname{Var} g_{S(k)\varphi} \leq \|g'_{S(k)\varphi}\|_{\sup} |I^{(k)}| \leq \mathcal{L}(\varphi)(2 + \log \|Q(k)\|) \frac{|I^{(k)}|}{\min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|}.$$

If $g_\varphi \neq 0$ then, by (5.3), we have $\mathrm{Var}(S(k)g_\varphi) \leq \mathrm{Var}\, g_\varphi$. As $g_{\varphi - g_\varphi} = 0$, by (5.5), it follows that

$$\begin{aligned}\mathcal{LV}(S(k)\varphi) &\leq \mathcal{L}(\varphi) + \mathrm{Var}(g_{S(k)(\varphi - g_\varphi)}) + \mathrm{Var}(S(k)g_\varphi)\\ &\leq \mathcal{L}(\varphi) + (2 + \log\|Q(k)\|)\frac{|I^{(k)}|}{\min_{\alpha\in\mathcal{A}}|I_\alpha^{(k)}|}\mathcal{L}(\varphi - g_\varphi) + \mathrm{Var}(g_\varphi)\\ &\leq (3 + \log\|Q(k)\|)\frac{|I^{(k)}|}{\min_{\alpha\in\mathcal{A}}|I_\alpha^{(k)}|}\mathcal{LV}(\varphi).\end{aligned}$$

□

Since by Definition of the Diophantine condition UDC (see (B1) in Definition 2 and Definition 3) the IETs obtained inducing on the subintervals $I^{(k)}$ are all *κ-balanced*, i.e. $|I^{(k)}| \leq \kappa \min_{\alpha\in\mathcal{A}} |I_\alpha^{(k)}|$, the conclusion of Lemma 5.11 immediately give the following Corollary.

Corollary 5.12. *Let T be an IET satisfying the UDC Then for all $0 \leq k \leq l$ and for every function $\varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ we have*

$$\mathcal{LV}(S(k,l)(\varphi)) \leq \kappa(3 + \log\|Q(k,l)\|)\mathcal{LV}(\varphi). \tag{5.15}$$

6. Correction operators

This section contains the statement and the proof of the key technical result of the paper (Theorem 6.1 below), which we now motivate and then state.

6.1. **Correction operator for cocycles with logarithmic singularities.**

Let φ be a function with logarithmic singularities and T an IET satisfying the Keane condition. Let $S(k)\varphi$ be a sequence of special Birkhoff sums obtained by renormalization, see § 5.1. Consider the sequence

$$\|S(k)\varphi\|_{L^1(I^{(k)})}/|I^{(k)}|, \qquad k \in \mathbb{N}, \tag{6.1}$$

of L^1-norms, renormalized by $|I^{(k)}|$. Notice that if $S(k)\varphi$ were bounded, the sequence would simply be controlled by the sequence of sup norms $\|S(k)\varphi\|_{L^\infty(I^{(k)})}$, $k \in \mathbb{N}$. Typically, the sequence in (6.1) grows exponentially with an exponent related to the Lyapunov exponents of the cocycle A_Y.

Our goal is to eliminate this growth, by *correcting* the function φ, namely by subtracting a piecewise constant function (constant on the continuity intervals of T). This piecewise constant function, which we call the *correction*, can be defined for IETs which satisfy the UDC and its values can be identified with a vector in $H(\pi)$. The correction vector will be given by a *correction operator* $\mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to H(\pi)$. We will call *correcting operator* the operator $P := I - \mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ which *performs* the correction, namely to φ associates the *corrected cocycle* $P(\varphi) = \varphi - \mathfrak{h}(\varphi)$ obtained subtracting the correction $\mathfrak{h}(\varphi)$. Under the assumption that T satisfies the UDC, for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, the correction $\mathfrak{h}(\varphi)$ will be such that the corrected function $P(\varphi) = \varphi - \mathfrak{h}(\varphi)$ produces a sequence

$$\|S(k) \circ P(\varphi)\|_{L^1(I^{(k)})}/|I^{(k)}|, \qquad k \in \mathbb{N}, \tag{6.2}$$

which now has sub-exponential growth. This will then be the starting point to show the existence of a full deviation spectrum for the L^1-norm (see § 7.2). Moreover, if additionally T satisfies the SUDC and $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ satisfies a stronger symmetry condition, $\mathcal{AS}(\varphi) = 0$, then the sequence (6.2) is bounded along a subsequence, and it will play a crucial role in the proof of ergodicity (see § 8).

6.1.1. *The main result on correction of logarithmic cocycles.*

The formal statement of the result that we are going to prove is the following.

Theorem 6.1 (Existence of a correction operator)**.** *Assume that $T = T_{(\pi,\lambda)}$ satisfies the UDC. There exists a bounded linear operator $\mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to H(\pi)$ such that for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathfrak{h}(\varphi) = 0$ we have*

$$\frac{\|S(k)\,\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq C\Big(C_k'(T)\|\varphi\|_{\mathcal{LV}} + \|Q_s(k)\|\frac{\|\varphi\|_{L^1(I)}}{|I|}\Big), \tag{6.3}$$

where $C_k'(T)$ is the Diophantine series defined in Definition 4. Furthermore, if additionally T satisfies the SUDC and $\mathcal{AS}(\varphi) = 0$ then

$$\frac{\|S(k)\,\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq C\Big(C_k(T)(\mathcal{LV}(\varphi) + \|\partial_{\pi^{(0)}}(\varphi)\|) + \|Q_s(k)\|\frac{\|\varphi\|_{L^1(I)}}{|I|}\Big), \tag{6.4}$$

where $C_k(T)$ is the other Diophantine series defined in Definition 4.

Combining Theorem 6.1 with the estimates on the Diophantine series given by Proposition 3.9 (see in particular (3.27)), we have the following corollary:

Corollary 6.2 (Subexponential growth of special Birkhoff sums of corrected cocycles)**.** *Given T and $\mathfrak{h}$ as in Theorem 6.1, for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathfrak{h}(\varphi) = 0$, we have*

$$\frac{\|S(k)\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} = O(e^{\tau k}).$$

Notice that, in virtue of the definition of the Diophantine series $C_k(T)$ and $C'_k(T)$, the control for the symmetric case given by (6.4) is finer than that given by (6.3) since $C'_k(T)$ has an additional term which is logarithmic in the matrix cocycle norms (which comes from the presence of $K'_l(T)$ instead of $K_l(T)$, see Definition 4).

Remark 6.3. More precisely, we will show in the proof of Theorem 6.1 that for any choice of a subspace $F \subset H(\pi)$ such that $F \oplus \Gamma_s(\pi) = H(\pi)$, where $\Gamma_s(\pi)$ is the stable space of $T = T_{(\pi,\lambda)}$, one can define a unique such operator $\mathfrak{h} = \mathfrak{h}_F$ such that $\mathfrak{h}_F(h) = h$ for any $h \in F$.

The proof of Theorem 6.1 will take the rest of this section. We first of all comment on the difficulties which motivate the change of strategy in comparison to [44] and [20] and give an outline of the main steps.

6.1.2. *Difficulties and outline of the proof.* The idea of *correction* as well of the strategy for proving of Theorem 6.1 are inspired by the seminal work by Marmi-Moussa-Yoccoz on the cohomological equation in [44] (see also [48]). As we already anticipated in the introduction, though, when considering functions with logarithmic singularities (or more in general BMO functions) and want to control the L^1-norm (which is the only one that we can controlled for functions with logarithmic singularities, which are unbounded), we need to modify substantially the original construction. The construction presented here is a modification of the construction that we introduced in [20] to prove an analogous result for IETs of *hyperbolic periodic type.* Working with almost every T, but requires again some major changes in the basic steps of construction. We comment here on the differences while giving an outline of the steps in the proof of Theorem 6.1.

First note that there is not an unique way to define a correction operator $\mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to H(\pi)$ with the desired properties (as in Theorem 6.1), since if we are given a function $\mathfrak{h}(\varphi)$ that satisfies the desired estimates (namely (6.3) and (6.4) in Theorem 6.1) and add an element from the stable space Γ_s, we get a new function that still satisfies the same estimates. On the other hand, if we compose with the projection $U : \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^{\mathcal{A}}/\Gamma_s$ to the quotient by the stable space, the *quotient operator*

$$\mathfrak{h}^U := U \circ \mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to H(\pi)/\Gamma_s$$

is uniquely defined and is the operator we are going to construct.

We will construct in fact a *sequence* of *correcting operators* with values in the quotient by the stable space, namely

$$P^{(k)} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)/\Gamma^{(k)}_s, \qquad k \in \mathbb{N}$$

(notice that if T satisfies the UDC the induced IET $T^{(k)}$ satisfies the UDC for every $k \geq 1$). For $k = 0$, the correcting operator $P^{(0)}$ will have the form $I - \mathfrak{h}^U$, where $\mathfrak{h}^U$ is the sought correction operator with values in the quotient. We want the sequence of operators $P^{(k)}, k \in \mathbb{N}$, to be *equivariant* under the action of the renormalization, i.e. to commute with the operation of taking special Birkhoff sums (see Lemma 6.7 for a precise statement).

The strategy to construct the sequence $P^{(k)}, k \in \mathbb{N}$ of equivariant correcting operators is the following:

(1) As first approximation of the correction operators, consider, for $k \in \mathbb{N}$, the mean value projections $\mathcal{M}^{(k)} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to \Gamma^{(k)}$, as defined in § 4.4, and the associated correcting operators $P^{(k)}_0 := I - \mathcal{M}^{(k)}$, $k \in \mathbb{N}$;

(2) The correcting operators $P^{(k)}_0$, $k \in \mathbb{N}$, are not equivariant and do not take values in the quotient. Let us hence modify them by subtracting a term $\Delta^{(k)}$ and composing with the projection $U^{(k)}$ to the quotient space $\Gamma^{(k)}/\Gamma^{(k)}_s$, namely consider, for each $k \in \mathbb{N}$, a operator of the form $P^{(k)} := U^{(k)} \circ P^{(k)}_0 - \Delta^{(k)}$;

(3) Following [44], one can see that for $P^{(0)}$ defined as in (2) to be equivariant, one needs to define $\Delta^{(0)}\varphi$ so that the modified correction operator $U \circ \mathcal{M}^{(0)} + \Delta^{(0)}$ is the limit (if it exists) of the sequence $U \circ Q(k)^{-1} \circ \mathcal{M}^{(k)} \circ S(k)(\varphi)$, which is obtained by '*bringing back*' the correction of $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$ at time k, namely of the function $S(k)(\varphi)$, to time 0 by applying $Q(k)^{-1}$;

(4) Show that the sequence in (3) converges, so that one can define the modification operator $\Delta^{(k)}$, then the correcting operator $P^{(k)} = U^{(k)} \circ P^{(k)}_0 - \Delta^{(k)}$ has the required covariance and growths properties.

Thus, to obtain the desired correction operator one has to show that the sequence

$$U \circ Q(k)^{-1} \circ \mathcal{M}^{(k)} \circ S(k)(\varphi) \in H(\pi^{(0)})/\Gamma^{(0)}_s, \qquad k \in \mathbb{N}$$

obtained in (3) converges for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$. Notice that when $\mathcal{M}^{(k)}$ takes values in $H(\pi^{(k)}) \subset \Gamma^{(k)}$ then $Q(k)^{-1}$ composed with the projection on Γ/Γ_s contracts exponentially and this allows to prove the convergence. In [44] and [20], though, the mean value projection $\mathcal{M}^{(k)}$, obtained taking mean values of the function over every exchanged interval (see (6.5) below) takes values also outside $H(\pi^{(k)})$. Therefore, the contraction argument does not apply. To circumvent this problem, in [20] we have used the the projection on Γ/Γ_{cs}, where Γ_{cs} is the *central stable* space. Unfortunately, though, this is not sufficient now, when we consider almost every IET.

One of the novelties in this part of the article in relations to the previous correction operators constructions is that we consider *initial corrections* $\mathcal{M}_H^{(k)}$ obtained by composing $\mathcal{M}^{(k)}$ with the projection $p_{H(\pi^{(k)})}$ onto the space $H(\pi^{(k)})$ (see § 6.2). In view of the boundary operator estimate given by Lemma 3.4 (see § 3.1.10), we can control the displacement between $\mathcal{M}_H^{(k)}$ and $\mathcal{M}^{(k)}$ in terms of the boundary operator $\partial_{\pi^{(k)}}$ (see § 6.2, in particular the proof of Lemma 6.4). It is starting from this modified preliminary correction operators in step (1) that allows to prove convergence and hence leads to a good definition of correction (and correcting) operators in the more general setting of this paper, but also requires proving a series of new inequalities and adding some new technical steps to the construction. The UDC is devised exactly in order to guarantee convergence of this series. In fact, to show that the series that gives $\Delta^{(k)}$ (which is written in (6.20)) converges, we will exploit the exponential contraction provided by the condition (UDC1) and (3.5).

The final part of the proof is to show that any correction operator $\mathfrak{h}$ defined choosing a representative $\mathfrak{h}(\varphi)$ for the equivalence class $\mathfrak{h}^U(\varphi)$ in $H(\pi)/\Gamma^s$ is such that $\|S(k)(\varphi - \mathfrak{h}(\varphi))\|_{L^1(I^{(k)})}/|I^{(k)}|$ has sub-exponential growth. This part follows quite closely the proof that we gave in [20], along the lines of [44].

6.2. **Preliminary corrections.** To define initial corrections, let us consider the linear operators on $\mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$, $k \in \mathbb{N}$, obtained by considering mean value-projections (which we defined in § 4.4)

$$\mathcal{M}^{(k)} : \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \Gamma^{(k)}, \quad \mathcal{M}^{(k)}\varphi = \sum_{\alpha\in\mathcal{A}} m(\varphi, I_\alpha^{(k)})\chi_{I_\alpha^{(k)}}. \tag{6.5}$$

6.2.1. *Initial corrections.* The sequence of initial corrections that we want to use is given by composing these mean value-projections with the projection onto the space $H(\pi^{(k)})$. Recall that $p_{H(\pi^{(k)})} : \Gamma^{(k)} \to H(\pi^{(k)})$ is the orthogonal projection on $H(\pi^{(k)})$.

Definition 8 (Initial corrections). Consider the operator

$$\mathcal{M}_H^{(k)} : \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to H(\pi^{(k)}), \qquad \mathcal{M}_H^{(k)} := p_{H(\pi^{(k)})} \circ \mathcal{M}^{(k)}.$$

Set the corresponding initial approximation of the correction operator to be

$$\begin{aligned} P_0^{(k)} : \quad & \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha ß\mathcal{A}} I_\alpha^{(k)}) \\ & \varphi \mapsto P_0^{(k)}\varphi := \varphi - \mathcal{M}_H^{(k)}\varphi. \end{aligned}$$

The following properties of the initial corrections follow almost directly from the estimates on mean average corrections that we proved in § 4.4 as preparatory work, combined with the control of the projection through the boundary operator (given by Lemma 3.4).

Lemma 6.4 (Initial correction estimates)**.** *There exists a positive constant C such that for every $k \in \mathbb{N}$, for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$,*

$$\|P_0^{(k)}\varphi\|_{L^1(I^{(k)})} \le C|I^{(k)}|\big(\log\|Q(k)\|\mathcal{AS}(\varphi) + \mathcal{LV}(\varphi) + \|\partial_{\pi^{(k)}}(\varphi)\|\big) \tag{6.6}$$

$$\|P_0^{(k)}\varphi\|_{L^1(I^{(k)})} \le 4dC|I^{(k)}|\log\big(2\kappa\|Q(k)\|\big)\|\varphi\|_{\mathcal{LV}} \tag{6.7}$$

$$\|\mathcal{M}_H^{(k)}\varphi\| \le \frac{\kappa\sqrt{d}}{|I^{(k)}|}\|\varphi\|_{L^1(I^{(k)})}. \tag{6.8}$$

Proof. To estimate $P_0^{(k)}$, we will compare $\mathcal{M}^{(k)}$ with $\mathcal{M}_H^{(k)}$, namely estimate

$$\|P_0^{(k)}\varphi\|_{L^1(I^{(k)})} = \|\varphi - \mathcal{M}_H^{(k)}\varphi\|_{L^1(I^{(k)})} \le \|\varphi - \mathcal{M}^{(k)}\varphi\|_{L^1(I^{(k)})} + \|\mathcal{M}^{(k)}\varphi - \mathcal{M}_H^{(k)}\varphi\|_{L^1(I^{(k)})} \tag{6.9}$$

Let us estimate separately the two terms in (6.9), namely the mean-value correcting operator and the difference of the mean value projections.

Estimating the mean-value correcting operator. By the construction of the mean projection operator (see (6.5) and the definition of m in § 4.3), we have

$$\|\mathcal{M}^{(k)}\varphi\|_{L^1(I^{(k)})} = \sum_{\alpha\in\mathcal{A}} |m(\varphi, I_\alpha^{(k)})||I_\alpha^{(k)}| = \sum_{\alpha\in\mathcal{A}} |\int_{I_\alpha^{(k)}} \varphi(x)\,dx| \le \|\varphi\|_{L^1(I^{(k)})} \tag{6.10}$$

and, by (4.8), we can therefore estimate the first term in (6.9) by

$$\|\varphi - \mathcal{M}^{(k)}\varphi\|_{L^1(I^{(k)})} \le 8|I^{(k)}|\mathcal{LV}(\varphi). \tag{6.11}$$

Estimating the difference of the mean value projections. To estimate the second term in (6.9), we recall that $p_{H(\pi^{(k)})}$, by Lemma 3.4, satisfies $\|h - p_{H(\pi^{(k)})}h\| \le C_{\mathcal{G}}\|\partial_{\pi^{(k)}}h\|$ for every $k \ge 0$ and $h \in \Gamma^{(k)}$. Thus,

$$\|\mathcal{M}^{(k)}\varphi - p_{H(\pi^{(k)})}\mathcal{M}^{(k)}\varphi\|_{L^1(I^{(k)})} \le |I^{(k)}|\ \|\mathcal{M}^{(k)}\varphi - p_{H(\pi^{(k)})}\mathcal{M}^{(k)}\varphi\| \le C_{\mathcal{G}}|I^{(k)}|\ \|\partial_{\pi^{(k)}}\mathcal{M}^{(k)}\varphi\|. \tag{6.12}$$

Moreover, by Proposition 4.6,

$$\|\partial_{\pi^{(k)}}\mathcal{M}^{(k)}\varphi\| \le \|\partial_{\pi^{(k)}}(\varphi)\| + \Big(1 + \log\frac{2\kappa}{|I^{(k)}|}\Big)\mathcal{AS}(\varphi) + 14d\kappa\mathcal{LV}(\varphi). \tag{6.13}$$

Proof of (6.6). Going back to (6.9) and combining the two separate estimates just proved, namely (6.11), (6.12) and (6.13), it follows that

$$\|P_0^{(k)}\varphi\|_{L^1(I^{(k)})} \le |I^{(k)}|\Big(C_{\mathcal{G}}\|\partial_{\pi^{(k)}}(\varphi)\| + C_{\mathcal{G}}\Big(1 + \log\frac{2\kappa}{|I^{(k)}|}\Big)\mathcal{AS}(\varphi) + (14d\kappa C_{\mathcal{G}} + 8)\mathcal{LV}(\varphi)\Big).$$

As, by (3.4), we have $|I^{(k)}|^{-1} \le \|Q(k)\|$, so we get $1 + \log(2\kappa/|I^{(k)}|) = O(\|Q(k)\|)$ which yields (6.6).

Proof of (6.7). Recall now that, by the estimate (4.18) of $\|\partial_{\pi^{(k)}}(\varphi)\|$ and balance, we have that $\|\partial_{\pi^{(k)}}(\varphi)\| \le 2d\log(2\kappa/|I^{(k)}|)\|\varphi\|_{\mathcal{LV}} \le 2d\log(2\kappa d\|Q(k)\|)\|\varphi\|_{\mathcal{LV}}$. Thus, as $\mathcal{AS}(\varphi) \le \mathcal{LV}(\varphi) \le \|\varphi\|_{\mathcal{LV}}$, it follows from (6.6) that

$$\begin{aligned}\|P_0^{(k)}\varphi\|_{L^1(I^{(k)})} &\le C|I^{(k)}|\left(\log\|Q(k)\| + 1 + 2d\log(2\kappa\|Q(k)\|)\right)\|\varphi\|_{\mathcal{LV}}\\ &\le 4dC|I^{(k)}|\log(2\kappa\|Q(k)\|)\|\varphi\|_{\mathcal{LV}}.\end{aligned}$$

This proves also (6.7).

Proof of (6.8). Finally, to prove (6.8), let us apply once more Lemma 3.4, which also gives that, for every $k \ge 0$ and $h \in \Gamma^{(k)}$, $\|p_{H(\pi^{(k)})}h\| \le \sqrt{d}\|h\|$. Using this combined with (6.10), we get

$$\|\mathcal{M}_H^{(k)}\varphi\| = \|p_{H(\pi^{(k)})}\mathcal{M}^{(k)}\varphi\| \le \sqrt{d}\|\mathcal{M}^{(k)}\varphi\| \le \frac{\kappa\sqrt{d}}{|I^{(k)}|}\|\mathcal{M}^{(k)}\varphi\|_{L^1(I^{(k)})} \le \frac{\kappa\sqrt{d}}{|I^{(k)}|}\|\varphi\|_{L^1(I^{(k)})}$$

which proves also (6.8) and concludes the proof. □

6.2.2. *The series bringing back the corrections.* We can now build the modification $\Delta^{(k)}$ as a series (see (6.15) below), obtained by quotienting and pulling back the preliminary corrections defined in the previous section.

Consider, for $k \in \mathbb{N}$, the projections $U^{(k)}$ on the quotient by the stable space, namely

$$U^{(k)} : \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)}) \to \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})/\Gamma_s^{(k)}.$$

Since $S(k,l)\Gamma_s^{(k)} = \Gamma_s^{(l)}$ and $S(k,l) : \Gamma^{(k)} \to \Gamma^{(l)}$ is invertible, the quotient linear transformation

$$S_\flat(k,l) : \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})/\Gamma_s^{(k)} \to \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(l)})/\Gamma_s^{(l)}$$

is well defined and $S_\flat(k,l) : \Gamma^{(k)}/\Gamma_s^{(k)} \to \Gamma^{(l)}/\Gamma_s^{(l)}$ is invertible. Moreover,

$$S_\flat(k,l) \circ U^{(k)}\varphi = U^{(l)} \circ S(k,l)\varphi \text{ for } \varphi \in \mathrm{LG}^{\mathrm{BV}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)}). \tag{6.14}$$

The following Lemma shows that our Diophantine Condition guarantees the convergence of the series (6.15) obtained bringing back the corrections and hence it can be used to define a modification operator $\Delta^{(k)}$. Furthermore, it provides estimates that show that the modification operator is bounded.

Lemma 6.5 (Convergence of the modification series)**.** *Suppose that T satisfies the UDC. For every function $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)})$, the following limit*

$$\Delta^{(k)}\varphi = \lim_{l\to\infty} U^{(k)} \circ S(k,l)^{-1} \circ \Big(S(k,l) \circ P_0^{(k)} - P_0^{(l)} \circ S(k,l)\Big)\,\varphi \tag{6.15}$$

exists in $H(\pi^{(k)})/\Gamma_s^{(k)}$ and

$$\|\Delta^{(k)}\varphi\| \le CK_k'(\mathcal{LV}(\varphi) + \|\partial_{\pi^{(k)}}(\varphi)\|). \tag{6.16}$$

Moreover,

$$\|\Delta^{(k)}(S(k)\varphi)\| \le CK_k'(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|) \textit{ for every } \varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha). \tag{6.17}$$

If additionally T satisfies the SUDC and $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ with $\mathcal{AS}(\varphi) = 0$ then for every $k \ge 1$ we have

$$\|\Delta^{(k)}(S(k)\varphi)\| \le CK_k(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|). \tag{6.18}$$

Let us first show that the Lemma implies that $\Delta^{(k)}$ is bounded.

Corollary 6.6 (Boundedness of the modification)**.** *For every $k \ge \mathbb{N}$, the operator $\Delta^{(k)} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha^{(k)}) \to H(\pi^{(k)})/\Gamma_s^{(k)}$ defined by* (6.15) *is bounded.*

Proof. In view of (6.16) and (4.18), for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ we have

$$\|\Delta^{(k)}\varphi\| \leq K_k'(1 + 2d\log(2\kappa d\|Q(k)\|))\|\varphi\|_{\mathcal{LV}}. \tag{6.19}$$

This shows that $\Delta^{(k)} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to H(\pi^{(k)})/\Gamma_s^{(k)}$ is bounded. □

The rest of this section is devoted to the proof of Lemma 6.5.

Proof of Lemma 6.5. Exploiting the telescopic nature of the series, calculations similar to those in [20] show that

$$\begin{aligned} U^{(k)} \circ S(k,l)^{-1} \circ \Big(S(k,l) \circ P_0^{(k)} - P_0^{(l)} \circ S(k,l)\Big) \\ = \sum_{k\leq r<l} (S_\flat(k, r+1))^{-1} \circ U^{(r+1)} \circ \mathcal{M}_H^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k,r). \end{aligned}$$

It follows that we need to prove the convergence of the series

$$\sum_{r\geq k} (S_\flat(k, r+1))^{-1} \circ U^{(r+1)} \circ \mathcal{M}_H^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k,r)\varphi \tag{6.20}$$

in $H(\pi^{(k)})/\Gamma_s^{(k)}$.

Convergence of the series and the estimate (6.16). For any $r \geq k$, using (6.8), (5.2), (6.6), we obtain

$$\begin{aligned} &\|\mathcal{M}_H^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k,r)\varphi\| \\ &\leq \frac{2\kappa\sqrt{d}}{|I^{(r+1)}|}\|S(r,r+1) \circ P_0^{(r)} \circ S(k,r)\varphi\|_{L^1(I^{(r+1)})} \\ &\leq \frac{2\kappa\sqrt{d}}{|I^{(r+1)}|}\|P_0^{(r)} \circ S(k,r)\varphi\|_{L^1(I^{(r)})} \\ &\leq C\frac{|I^{(r)}|}{|I^{(r+1)}|}\big(\mathcal{AS}(S(k,r)\varphi)\log\|Q(r)\| + \mathcal{LV}(S(k,r)\varphi) + \|\partial_{\pi^{(r)}}(S(k,r)\varphi)\|\big). \end{aligned}$$

By the invariance of $\mathcal{AS}$, $\mathcal{LV}$ and the boundary operator (see (5.15), (5.6) in Corollary 5.3, (5.7)), (3.4) and (4.5) consecutively, we have

$$\begin{aligned} &\|\mathcal{M}_H^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k,r)\varphi\| \\ &\leq C\frac{|I^{(r)}|}{|I^{(r+1)}|}(\mathcal{LV}(S(k,r)\varphi) + \|\partial_{\pi^{(r)}}(S(k,r)\varphi)\| + \mathcal{AS}(S(k,r)\varphi)\log\|Q(r)\|) \\ &\leq C\frac{|I^{(r)}|}{|I^{(r+1)}|}(\kappa(3 + \log\|Q(k,r)\|)\mathcal{LV}(\varphi) + \|\partial_{\pi^{(k)}}(\varphi)\| + \mathcal{AS}(\varphi)\log\|Q(r)\|) \\ &\leq C'\|Z(r+1)\|(\mathcal{LV}(\varphi) + \|\partial_{\pi^{(k)}}(\varphi)\|)\log\|Q(r)\|. \end{aligned}$$

In view of (3.5), for $0 \leq k < l$ and $h \in H(\pi^{(l)})$ we have

$$\|(S_\flat(k,l))^{-1} \circ U^{(l)}(h)\| \leq \|Q_s(k,l)\|\|U^{(l)}(h)\| \leq \|Q_s(k,l)\|\|h\|. \tag{6.21}$$

Since $\mathcal{M}_H^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k,r)\varphi \in H(\pi^{(r+1)})$, by (6.21), the norm of the r-th element of the series (6.20) is bounded from above by

$$C'\|Q_s(k, r+1)\|\|Z(r+1)\|(\mathcal{LV}(\varphi) + \|\partial_{\pi^{(k)}}(\varphi)\|)\log\|Q(r)\|.$$

Since T satisfies the UDC, by Proposition 3.9, the series

$$\sum_{r\geq k}\|Q_s(k, r+1)\|\|Z(r+1)\|\log\|Q(r)\|$$

is convergent and its sum is K_k'. As $\Delta^{(k)}\varphi$ is the sum of the series (6.20), it follows that the operator $\Delta^{(k)}$ is well defined and (6.16) holds.

The estimates (6.17). If $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ then we can repeat the above arguments for $S(k)\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ instead of φ. As

$$\begin{gathered} \mathcal{LV}(S(k,r)(S(k)\varphi)) \leq C\log\|Q(r)\|\mathcal{LV}(\varphi), \\ \|\partial_{\pi^{(r)}}(S(k,r)(S(k)\varphi))\| = \|\partial_\pi(\varphi)\|,\ \mathcal{AS}(S(k,r)(S(k)\varphi)) = \mathcal{AS}(\varphi), \end{gathered}$$

now the norm of the r-th element of the series (6.20) where φ is replaced by $S(k)\varphi$, is bounded from above by

$$C'\|Q_s(k, r+1)\|\|Z(r+1)\|(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|)\log\|Q(r)\|.$$

This gives also (6.17).

Symmetric singularities estimates. Now suppose that T satisfies the SUDC and $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathcal{AS}(\varphi) = 0$. Using (5.12) and reasoning similar to the above we obtain

$$\begin{aligned}\|P_0^{(r)} \circ S(k,r)(S(k)\varphi)\|_{L^1(I^{(r)})} &= \|P_0^{(r)}(S(r)\varphi)\|_{L^1(I^{(r)})} \leq C|I^{(r)}|(\mathcal{LV}(S(r)\varphi) + \|\partial_{\pi^{(r)}}(S(r)\varphi)\|)\\ &\leq C'\|Z(r+1)\||I^{(r+1)}|(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|).\end{aligned}$$

Thus

$$\begin{aligned}&\|(S_\flat(k,r+1))^{-1} \circ U^{(r+1)} \circ \mathcal{M}_H^{(r+1)} \circ S(r,r+1) \circ P_0^{(r)} \circ S(k,r)(S(k)\varphi)\|\\ &\leq C\|Q_s(k,r+1)\|\|Z(r+1)\|(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|).\end{aligned}$$

This gives (6.18). □

6.2.3. *The equivariant correction operators.* Consider now the operator

$$P^{(k)} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})/\Gamma_s^{(k)}$$

given by $P^{(k)} = U^{(k)} \circ P_0^{(k)} - \Delta^{(k)}$. As the operators $U^{(k)}$ and $P_0^{(k)}$ (see (6.7)) are bounded linear operators, $P^{(k)}$ is also linear and bounded when $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})/\Gamma_s^{(k)}$ is equipped with the $L^1(I^{(k)})/\Gamma_s^{(k)}$ norm. We will now show that this modified correcting operator satisfies the sough equivariance property, i.e. *commutes* with the operation of considering special Birkhoff sums.

Lemma 6.7 (Equivariance)**.** *Suppose that T satisfies the UDC. For all $0 \leq k \leq l$ we have*

$$S_\flat(k,l) \circ P^{(k)} = P^{(l)} \circ S(k,l). \tag{6.22}$$

Moreover, for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ we have

$$\frac{1}{|I^{(k)}|}\|P^{(k)}(S(k)\varphi)\|_{L^1(I^{(k)})/\Gamma_s^{(k)}} \leq \Theta_k(\varphi) := CK_k'\|\varphi\|_{\mathcal{LV}}. \tag{6.23}$$

If additionally T satisfies the SUDC and $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathcal{AS}(\varphi) = 0$ then (6.23) *holds with*

$$\Theta_k(\varphi) := CK_k(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|).$$

Proof. The condition (6.22) is a direct consequence of the definition of $P^{(k)}$. Its proof run along similar lines as the proof of the first part of Lemma 4.2 in [20].
In view of $\|U^{(k)}\| = 1$, (6.6), (6.17), (5.14) and (5.7) we get

$$\|P^{(k)}(S(k)\varphi)\|_{L^1(I^{(k)})/\Gamma_s^{(k)}} \leq \|P_0^{(k)}(S(k)\varphi)\|_{L^1(I^{(k)})} + |I^{(k)}|\|\Delta^{(k)}(S(k)\varphi)\| \leq C|I^{(k)}|K_k'(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|).$$

Moreover, using (6.7) and (6.19) instead of (6.6) and (6.17), we also have

$$\|P^{(k)}(S(k)\varphi)\|_{L^1(I^{(k)})/\Gamma_s^{(k)}} \leq C|I^{(k)}|K_k'\|\varphi\|_{\mathcal{LV}},$$

which give (6.23).

Symmetric singularities case. Suppose that T satisfies the SUDC and $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathcal{AS}(\varphi) = 0$. Then, using (6.18) and (5.12) instead of (6.16) and (5.14), we get (6.23) with $\Theta_k(\varphi) = CK_k(\mathcal{LV}(\varphi) + \|\partial_\pi(\varphi)\|)$. □

6.3. **Proof of Theorem 6.1.** Now that we have build the correcting operator $P^{(0)}$ with values in the space $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})/\Gamma_s^{(0)}$ and the desired equivariance properties (see Lemma 6.7, we want to check that any choice of representative for the equivalence class $P^{(0)}\varphi$ satisfies the desired growth estimates and then to lift $P^{(0)}$ to an operator $I - \mathfrak{h}$ with values in $\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$. We first prove a Lemma that shows that any choice of representative of the equivalence class $P^{(0)}(\varphi)$ satisfies the desired estimates hold (see Lemma 6.8 and in particular the estimates in (ii)) and then use it to show that the correction is uniquely defined (see Corollary 6.9). The proof of Theorem 6.1 then follows easily from this Lemma 6.8 and Corollary 6.9 and is given at the end of the section.

Recall that we defined the equivariant correction operator $P^{(0)}$ by setting $P^{(0)} = U^{(0)} \circ P_0^{(0)} - \Delta^{(0)}$. We say that a map $\widehat{\varphi} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$ is a *correction* of φ if it is a *representative of the corrected equivalence class* $P^{(0)}\varphi$, i.e. $U^{(0)}(\widehat{\varphi}) = P^{(0)}(\varphi)$. With this in mind, the following Lemma shows that any correction of φ satisfies the desired estimates on the growth of Birkhoff sums. The constants C_k and C_k' which appear in the estimates of Birkhoff sums of corrected functions (see part (ii) of the Lemma below) are given by the *Diophantine series* $C_k(T)$ and $C_k'(T)$ which we defined for any $k \in \mathbb{N}$ in § 4 and showed that they converge and hence are well defined under the assumption that T satisfies the UDC or SUDC.

Lemma 6.8 (Birkhoff sums estimates for corrected functions)**.** *Suppose that T satisfies the UDC. Assume that $\varphi, \widehat{\varphi} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(0)})$ and that $U^{(0)}\widehat{\varphi} = P^{(0)}\varphi$. Then:*

(i) $\widehat{\varphi} - \varphi \in H(\pi^{(0)})$.

(ii) For any $k \geq 1$ we have

$$\frac{\|S(k)(\widehat{\varphi})\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq C\Big(C'_k\|\varphi\|_{\mathcal{LV}} + C''_k\frac{\|\widehat{\varphi}\|_{L^1(I^{(0)})}}{|I^{(0)}|}\Big) \tag{6.24}$$

and if T satisfies the SUDC and $\mathcal{AS}(\varphi) = 0$ then

$$\frac{\|S(k)(\widehat{\varphi})\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq C\Big(C_k\big(\mathcal{LV}(\varphi) + \|\partial_{\pi^{(0)}}(\varphi)\|\big) + C''_k\frac{\|\widehat{\varphi}\|_{L^1(I^{(0)})}}{|I^{(0)}|}\Big) \tag{6.25}$$

with $C_k := C_k(T)$, $C'_k := C'_k(T)$ (refer to § 4 for the definition of the Diophantine series C_k and C'_k) and $C''_k := \|Q_s(k)\|$.

The Lemma shows that every correction of φ is of the form $\varphi - h$ with $h \in H(\pi^{(0)})$. Let us first show that the Lemma also implies that the correction h is uniquely defined, once we fix a complement to $\Gamma_s^{(0)}$ in $H(\pi^{(0)})$.

Corollary 6.9 (Uniqueness of the correction)**.** *Fix a subspace $F \subset H(\pi^{(0)})$ such that $F \oplus \Gamma_s^{(0)} = H(\pi^{(0)})$. Suppose that $h_1, h_2 \in F$ are two vectors such that*

$$U^{(0)}(\varphi - h_1) = U^{(0)}(\varphi - h_2) = P^{(0)}\varphi.$$

Then $h_1 = h_2$.

Proof. In view of (6.24) of Lemma 6.8 combined with (3.27), we have

$$\limsup_{k\to+\infty} \frac{\log \frac{\|S(k)(\varphi - h_i)\|_{L^1(I^{(k)})}}{|I^{(k)}|}}{k} \leq 0 \text{ for } i = 1, 2.$$

Thus

$$\limsup_{k\to+\infty} \frac{\log\|Q(k)(h_1 - h_2)\|}{k} \leq 0.$$

As $h_1 - h_2 \in H(\pi^{(0)})$, by the condition (O) in Definition 3, it follows that $h_1 - h_2 \in \Gamma_s^{(0)}$. Since $h_1 - h_2 \in F$ and $\Gamma_s^{(0)} \cap F = \{0\}$, we have $h_1 = h_2$. □

Let us now prove the Lemma.

Proof of Lemma 6.8. Since by definition of the operators

$$U^{(0)}\widehat{\varphi} = P^{(0)}\varphi = U^{(0)} \circ P_0^{(0)}\varphi - \Delta^{(0)}\varphi = U^{(0)}\varphi - U^{(0)} \circ \mathcal{M}_H^{(0)}\varphi - \Delta^{(0)}\varphi,$$

we have

$$U^{(0)}(\varphi - \widehat{\varphi}) = U^{(0)} \circ \mathcal{M}_H^{(0)}\varphi + \Delta^{(0)}\varphi \in H(\pi^{(0)})/\Gamma_s^{(0)}.$$

Therefore

$$\varphi - \widehat{\varphi} \in H(\pi^{(0)}) + \Gamma_s^{(0)} \subset H(\pi^{(0)}). \tag{6.26}$$

In view of (6.14) and (6.22),

$$U^{(k)} \circ S(k)\widehat{\varphi} = S_\flat(k) \circ U^{(0)}\widehat{\varphi} = S_\flat(k) \circ P^{(0)}\varphi = P^{(k)} \circ S(k)\varphi.$$

Therefore, from (6.23), we have

$$\|U^{(k)} \circ S(k)\widehat{\varphi}\|_{L^1(I^{(k)})/\Gamma_s^{(k)}} = \|P^{(k)}(S(k)\varphi)\|_{L^1(I^{(k)})/\Gamma_s^{(k)}}|I^{(k)}| \leq \Theta_k(\varphi).$$

It follows from the definition of $\|\cdot\|_{L^1(I^{(k)})/\Gamma_s^{(k)}}$ on the quotient space that for every $k \geq 0$ there exists $\varphi_k \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and $s_k \in \Gamma_s^{(k)}$ such that

$$S(k)\widehat{\varphi} = \varphi_k + s_k \text{ and } \frac{\|\varphi_k\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq \Theta_k(\varphi). \tag{6.27}$$

Next note that

$$\varphi_{k+1} + s_{k+1} = S(k+1)\widehat{\varphi} = S(k, k+1)S(k)\widehat{\varphi} = S(k, k+1)\varphi_k + Q(k, k+1)s_k, \tag{6.28}$$

so setting $\Delta s_{k+1} = s_{k+1} - Z(k+1)s_k$ ($\Delta s_0 = s_0$) we have

$$\Delta s_{k+1} = -\varphi_{k+1} + S(k, k+1)\varphi_k.$$

Moreover, by (5.2), for $k \geq 0$,

$$\begin{aligned}\|\Delta s_{k+1}\| &\leq \frac{\kappa}{|I^{(k+1)}|}\|\Delta s_{k+1}\|_{L^1(I^{(k+1)})} = \frac{\kappa}{|I^{(k+1)}|}\|\varphi_{k+1} - S(k,k+1)\varphi_k\|_{L^1(I^{(k+1)})} \\ &\leq \frac{\kappa}{|I^{(k+1)}|}\big(\|\varphi_{k+1}\|_{L^1(I^{(k+1)})} + \|S(k,k+1)\varphi_k\|_{L^1(I^{(k+1)})}\big) \\ &\leq \kappa\Big(\frac{\|\varphi_{k+1}\|_{L^1(I^{(k+1)})}}{|I^{(k+1)}|} + \frac{|I^{(k)}|}{|I^{(k+1)}|}\frac{\|\varphi_k\|_{L^1(I^{(k)})}}{|I^{(k)}|}\Big).\end{aligned}$$

Next, by (6.27) and (3.4), it follows that

$$\|\Delta s_{k+1}\| \leq \kappa\big(\|Z(k+1)\|\Theta_k(\varphi) + \Theta_{k+1}(\varphi)\big) \text{ for } k \geq 0$$

and

$$\|\Delta s_0\| \leq \kappa\frac{\|s_0\|_{L^1(I^{(0)})}}{|I^{(0)}|} = \kappa\frac{\|\widehat{\varphi} - \varphi_0\|_{L^1(I^{(0)})}}{|I^{(0)}|} \leq \kappa\frac{\|\widehat{\varphi}\|_{L^1(I^{(0)})}}{|I^{(0)}|} + \kappa\Theta_0(\varphi).$$

Since $s_k = \sum_{0\leq l\leq k} Q(l,k)\Delta s_l$ and $\Delta s_l \in \Gamma_s^{(l)}$, setting $\Theta_{-1} := 0$, we have

$$\begin{aligned}\|s_k\| &\leq \sum_{0\leq l\leq k}\|Q(l,k)\Delta s_l\| \leq \sum_{0\leq l\leq k}\|Q_s(l,k)\|\|\Delta s_l\| \\ &\leq \kappa\sum_{0\leq l\leq k}\|Q_s(l,k)\|(\Theta_l(\varphi) + \|Z(l)\|\Theta_{l-1}(\varphi)) + \kappa\|Q_s(k)\|\frac{\|\widehat{\varphi}\|_{L^1(I^{(0)})}}{|I^{(0)}|}.\end{aligned}$$

In view of (6.27) and taking $\Theta_k(\varphi) = CK'_k\|\varphi\|_{\mathcal{LV}}$, it follows that for $k \geq 1$,

$$\frac{\|S(k)\widehat{\varphi}\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq \frac{\|\varphi_k\|_{L^1(I^{(k)})}}{|I^{(k)}|} + \|s_k\| \leq d\kappa C\Big(C'_k\|\varphi\|_{\mathcal{LV}} + C''_k\frac{\|\varphi\|_{L^1(I^{(0)})}}{|I^{(0)}|}\Big).$$

If T satisfies the SUDC and $\mathcal{AS}(\varphi) = 0$ then the same argument applied to $\Theta_k(\varphi) = CK_k(\mathcal{LV}(\varphi) + \|\partial_{\pi^{(0)}}(\varphi)\|)$ shows also (6.25). □

We have now all the elements to conclude the proof of Theorem 6.1.

Proof of Theorem 6.1. Fix a subspace $F \subset H(\pi^{(0)})$ such that $F \oplus \Gamma_s^{(0)} = H(\pi^{(0)})$. Choose any $\widehat{\varphi} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $U^{(0)}(\widehat{\varphi}) = P^{(0)}\varphi$. By (i) of Lemma 6.8, $\widehat{\varphi} - \varphi \in H(\pi^{(0)})$. Therefore, there exist $h \in F$ and $h' \in \Gamma_s^{(0)}$ such that $\varphi - h = \widehat{\varphi} + h'$. As $U^{(0)}(\widehat{\varphi}) = P^{(0)}\varphi$, it follows that

$$U^{(0)}(\varphi - h) = U^{(0)}(\widehat{\varphi} + h') = U^{(0)}(\widehat{\varphi}) = P^{(0)}\varphi.$$

By Corollary 6.9, for every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ there exists a unique $h = h(\varphi) \in F$ such that $U^{(0)}(\varphi - h) = P^{(0)}\varphi$. Thus, there exists a unique linear operator $\mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to F$ (the *correction operator*) such that

$$U^{(0)}(\varphi - \mathfrak{h}(\varphi)) = P^{(0)}(\varphi). \tag{6.29}$$

As the operator $P^{(0)} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)/\Gamma_s^{(0)}$ is bounded, by the closed graph theorem, the operator $\mathfrak{h}$ is also bounded. Indeed, if $\varphi_n \to \varphi$ in LG and $\mathfrak{h}(\varphi_n) \to h$ in F then have both

$$\begin{aligned}P^{(0)}\varphi_n &\to P^{(0)}\varphi = U^{(0)}(\varphi - \mathfrak{h}(\varphi)), \\ P^{(0)}\varphi_n &= U^{(0)}(\varphi_n - \mathfrak{h}(\varphi_n)) \to U^{(0)}(\varphi - h).\end{aligned}$$

It follows that $\mathfrak{h}(\varphi) - h \in F$ and at the same time $\mathfrak{h}(\varphi) - h \in \Gamma_s^{(0)}$, so $h = \mathfrak{h}(\varphi)$. Since the vector norm and the L^1-norm are equivalent on $\Gamma^{(0)}$, we get that the operator is bounded.

Suppose now that $\mathfrak{h}(\varphi) = 0$. Then

$$U^{(0)}(\varphi) = U^{(0)}(\varphi - \mathfrak{h}(\varphi)) = P^{(0)}(\varphi).$$

Therefore, (6.3) and (6.4) follow directly from (6.24) and (6.25) of the part (ii) of Lemma 6.8 respectively. This concludes the proof and proves as well the statement of Remark 6.3. □

7. Deviations of Birkhoff sums and integrals

In this section we prove the main results on the deviation spectrum of locally Hamiltonian flows, by first reducing the study of integrals along a locally Hamiltonian flow to the study of Birkhoff sums (see § 7.1), then exploiting the correction operator built in § 6 to build (in the spirit of Bufetov functionals and Bufetov work [6]) the cocycles which correspond to pure power behaviour, see § 7.2.

7.1. **Estimates of Birkhoff integrals through Birkhoff sums.** In this section we provide effective estimate for the growth of Birkhoff integrals (Proposition 7.7), which can be applied when the roof function g is unbounded. We first exploit the special flow representation of the flow as a suspension flow over an IET under a roof function with logarithmic singularities (refer to § 2.3) to reduce to estimates of Birkhoff sums, see § 7.1.1. We then exploit a standard decomposition of Birkhoff sums in special Birkhoff sums, see § 7.1.2. The estimates relies on the speed of decay of the tails of g. This crucial new ingredient is explained in § 7.1.3. The main result of this section is then the estimate given by Proposition 7.7 in § 7.1.4.

7.1.1. *Reduction of integrals along the flow to Birkhoff sums.* Let $T : I \to I$ be an ergodic IET and let $g : I \to \mathbb{R}_{>0} \cup \{+\infty\}$ be an integrable function such that $\underline{g} = \inf_{x\in I} g(x) > 0$. Following § 2.2.2, we denote by $T^g_{\mathbb{R}} : I^g \to I^g$ the special flow over T under the roof g. For every integrable function $f : I^g \to \mathbb{R}$ let $\varphi_f : I \to \mathbb{R}$ be given by $\varphi_f(x) = \int_0^{g(x)} f(x,r)\,dr$. By Fubini's theorem, φ_f is well defined for a.e. $x \in I$, is integrable and

$$\int_I \varphi_f(x)\,dx = \int_{I^g} f(x,r)\,dx\,dr.$$

For every $(x,r) \in I^g$ and $s > 0$ denote by $n(x,r,s) \geq 0$ the number of times the orbit segment $\{T^g_t(x,r) : t \in [0,s]\}$ crosses the interval I (identified with $I \times \{0\}$), i.e. the unique non-negative integer number such that

$$g^{(n(x,r,s))}(x) \leq r + s < g^{(n(x,r,s)+1)}(x). \tag{7.1}$$

Then $0 \leq n(x,r,s) \leq s/\underline{g} + 1$.

For every $c \geq \underline{g}$, let $I_c \subset I$ be the level set defined by $g(x) \leq c$ for every $x \in I_c$. Moreover, for every $s \geq 0$ let

$$A^s_c := \{(x,r) \in I^g : x \in I_c\} \setminus \{T^g_{-t}(x,0) : x \in I \setminus I_c,\ 0 \leq t \leq s\} \subset I^g. \tag{7.2}$$

The following elementary Lemma relates the Birkhoff integrals of f for the flow $T^g_{\mathbb{R}}$ with the Birkhoff sums of φ_f for the IET T.

Lemma 7.1. *Suppose that $f : I^g \to \mathbb{R}$ is bounded. For every $s > 0$ and $c \geq \underline{g}$ if $(x,r) \in A^s_c$ then $T^i x \in I_c$ for all $0 \leq i \leq n(x,r,s)$, and*

$$\Big|\int_0^s f(T^g_t(x,r))\,dt\Big| \leq |\varphi_f^{(n(x,r,s))}(x)| + 2c\|f\|_{L^\infty}. \tag{7.3}$$

Proof. For every $(x,r) \in A^s_c$ we decompose the orbit segment $\{T^g_t(x,r) : t \in [0,s]\}$ into $n(x,r,s)+1$-pieces using its meeting points with $I \times \{0\} \subset I^g$, i.e. along crossing times

$$0 < t_1 < \ldots < t_n < s, \text{ where } n := n(x,r,s) \text{ and } t_i := g^{(i)}(x) - r \text{ for } 1 \leq i \leq n.$$

Then $T^g_{t_i}(x,r) = (T^i x, 0)$ for $0 \leq i \leq n$, with $t_0 := -r$. As $(x,r) \in A^s_c$, it follows that $g(T^i x) \leq c$ for $0 \leq i \leq n$, which proves the first part of the Lemma. As $t_{i+1} - t_i = g^{(i+1)}(x) - g^{(i)}(x) = g(T^i x)$, according to the decomposition we obtain

$$\begin{aligned}
\int_0^s f(T^g_t(x,r))\,dt &= \int_0^{t_0} f(T^g_t(x,r))\,dt + \sum_{0\leq j<n} \int_{t_j}^{t_{j+1}} f(T^g_t(x,r))\,dt + \int_{t_n}^s f(T^g_t(x,r))\,dt \\
&= \sum_{0\leq j<n} \int_0^{g(T^j x)} f(T^j x,t)\,dt - \int_0^r f(x,t)\,dt + \int_0^{s-t_n} f(T^n x,t)\,dt \\
&= \varphi_f^{(n)}(x) - \int_0^r f(x,t)\,dt + \int_0^{s-t_n} f(T^n x,t)\,dt.
\end{aligned}$$

Since $r < g(x) \leq c$ and $s - t_n < g(T^n) \leq c$, we also have

$$\Big|\int_0^r f(x,t)\,dt\Big| \leq \int_0^{g(x)} |f(x,t)|\,dt \leq c\|f\|_{L^\infty}$$

and

$$\Big|\int_0^{s-t_n} f(T^n x,t)\,dt\Big| \leq \int_0^{g(T^n x)} |f(T^n x,t)|\,dt \leq c\|f\|_{L^\infty}.$$

Therefore

$$\Big|\int_0^s f(T^g_t(x,r))\,dt\Big| \leq \big|\varphi_f^{(n(x,r,s))}(x)\big| + 2c\|f\|_{L^\infty}$$

for every $(x,r) \in A^s_c$. □

7.1.2. *Decomposition of Birkhoff sums in special Birkhoff sums.* In this subsection we estimate $\varphi^{(n)}(x)$ by decomposing the sum into special Birkhoff sums introduced by Zorich in [71]. Let $T : I \to I$ be an arbitrary IET satisfying Keane's condition. For every $x \in I$ and $n \geq 0$ set

$$m(x, n) = m(x, n, T) := \max\left\{l \geq 0 : \#\{0 \leq k \leq n : T^k x \in I^{(l)}\} \geq 2\right\}.$$

Proposition 7.2 (see [71] or [67])**.** *For every $x \in I$ and $n > 0$ we have*

$$\min_{\alpha\in\mathcal{A}} Q_\alpha(m) \leq n \leq d \max_{\alpha\in\mathcal{A}} Q_\alpha(m + 1) = d\|Q(m + 1)\|, \text{ where } m = m(x, n).$$

Since the sequence $\left(\min_{\alpha\in\mathcal{A}} Q_\alpha(m)\right)_{m\geq 0}$ increases to the infinity

$$m(n) = m(n, T) := \max\{m(x, n) : x \in I\}$$

is well defined. If T additionally satisfies the UDC then, by (UDC3) and (3.17), for every $\tau > 0$ we have

$$e^{\lambda_1 m(n)} \leq O(\|Q(m(n))\|) \leq O\Big(\min_{\alpha\in\mathcal{A}} Q_\alpha(m(n))^{1+\tau}\Big) = O(n^{1+\tau}). \tag{7.4}$$

Proposition 7.3. *For every $s > 0$ and $c \geq \underline{g}$ if $(x, r) \in A^s_c$ then*

$$\left|\varphi_f^{(n(x,r,s))}(x)\right| \leq 2 \sum_{k=0}^{m(n(x,r,s))} \|Z(k + 1)\|\|S(k)\varphi_f\|_{L^\infty(I^{(k)}(c))}, \tag{7.5}$$

with

$$I^{(k)}(c) := \bigcup_{\alpha\in\mathcal{A}} \{x \in I^{(k)}_\alpha : \forall_{0\leq j<Q_\alpha(k)} T^j x \in I_c\}.$$

Proof. Fix $s > 0$ and $c > 0$. For each point $(x, r) \in I^g$ we will decompose the orbit segment

$$x, Tx, \ldots, T^{n-1}x \text{ with } n := n(x, r, s)$$

into segments. Let $m := m(x, n)$, so $I^{(m)}$ is hit by the the orbit segment at least twice and $I^{(m+1)}$ at most once. For each $0 \leq k \leq m$ let

$$n^+_k = \min\{j \geq 0 : T^j x \in I^{(k)}\}, \quad n^-_k = \min\{j \geq 1 : T^{n-j} x \in I^{(k)}\}.$$

For $0 \leq k < m$ we also have

$$T^{n^+_{k+1}}x = (T^{(k)})^{b^+_k} T^{n^+_k} x \text{ and } T^{n-n^-_{k+1}}x = (T^{(k)})^{-b^-_k} T^{n-n_k} x$$

with

$$0 \leq b^+_k, b^-_k < \|Z(k + 1)\|. \tag{7.6}$$

Moreover,

$$(T^{(m)})^{b_m} T^{n^+_m} x = T^{n-n^-_m} x \text{ with } 1 \leq b_m \leq \|Z(m + 1)\|. \tag{7.7}$$

Here $T^{n^+_m}x$, $T^{n-n^-_m}x$ are the first and the last visit of the orbit segment in $I^{(m)}$. Thus

$$\begin{aligned}\varphi_f^{(n)}(x) = &\sum_{k=0}^{m-1}\sum_{j=0}^{b^+_k-1} (S(k)\varphi_f)((T^{(k)})^j T^{n^+_k} x) + \sum_{j=0}^{b_m-1} (S(m)\varphi_f)((T^{(m)})^j T^{n^+_m} x)\\ &+ \sum_{k=0}^{m-1}\sum_{j=0}^{b^-_k-1} (S(k)\varphi_f)((T^{(k)})^j T^{n-n^-_{k+1}} x).\end{aligned}$$

If $(x, r) \in A^c_s$, then, by the first part of Lemma 7.1, $T^l x \in I_c$ for all $0 \leq l \leq n$. Hence

$$(T^{(k)})^j T^{n^+_k} x, (T^{(k)})^j T^{n-n^-_{k+1}} x \in I^{(k)}(c).$$

In view of (7.6) and (7.7), it follows that

$$|\varphi_f^{(n)}(x)| \leq 2 \sum_{k=0}^{m} \|Z(k + 1)\|\|S(k)\varphi_f\|_{L^\infty(I^{(k)}(c))},$$

which proves (7.5). □

7.1.3. *Control of the tail behaviour.* Let $g : I \to \mathbb{R}_{>0} \cup \{+\infty\}$ be an integrable roof map with $\underline{g} = \min_{x\in I} g(x) > 0$. Suppose that for every $s \geq \underline{g}$ we have a subset $I_s \subset I$ such that $g(x) \leq s$ for $x \in I_s$. Let us consider the map $\xi : [\underline{g}, +\infty) \to \mathbb{R}_{\geq 0}$ given by

$$\xi(s) := Leb(I \setminus I_s). \tag{7.8}$$

Denote by $F_g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ the *tail distribution function* of g, i.e.

$$F_g(s) := Leb(\{x \in I : g(x) > s\}) \text{ for } s \geq 0.$$

By definition,

$$\{x \in I : g(x) > s\} \subset I \setminus I_s \text{ and } F_g(s) \leq \xi(s) \text{ for } s \geq \underline{g}. \tag{7.9}$$

Lemma 7.4. *Suppose that $\xi : [\underline{g}, +\infty) \to \mathbb{R}_{\geq 0}$ is decreasing integrable and of class C^1 map with $\lim_{s\to+\infty} s\xi(s) = 0$. Let us consider $\Xi : [\underline{g}, +\infty) \to \mathbb{R}_{\geq 0}$ be given by $\Xi(s) = \int_s^{+\infty} \xi(t)\, dt$ for $s \geq \underline{g}$. Then for every $s > 0$ and $c \geq \underline{g}$ we have*

$$Leb(I^g \setminus A^s_c) \leq s\xi(c) + 2c\xi(c) + \Xi(c). \tag{7.10}$$

Proof. By the definition of Ξ and (7.9), using integration by part we have

$$\int_{\{x\in I:g(x)\geq c\}} g(x)dx = -\int_c^\infty t\, dF_g(t) = cF_g(c) + \int_c^\infty F_g(t)\, dt \leq c\xi(c) + \Xi(c).$$

Therefore

$$\int_{I\setminus I_c} g(x)dx \leq \int_{\{x\in I:g(x)\geq c\}} g(x)dx + \int_{\{x\in I\setminus I_c:g(x)\leq c\}} g(x)dx \leq 2c\xi(c) + \Xi(c).$$

It follows that for every $c \geq \underline{g}$ and $s \geq 0$ we have

$$Leb(I^g \setminus A^s_c) \leq \int_{I\setminus I_c} g(x)dx + sLeb(I \setminus I_c) = s\xi(c) + 2c\xi(c) + \Xi(c),$$

which completes the proof □

Remark 7.5. Note that, by definition, Ξ is a decreasing C^2-map and $\lim_{s\to+\infty} \Xi(s) = 0$.

Remark 7.6. Suppose that the roof function $g \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. Then there exist two positive constants $C, b > 0$ such that for every $s \geq \underline{g}$ we have

$$g(x) \leq s \text{ for all } x \in \bigcup_{\alpha\in\mathcal{A}} [l_\alpha + Ce^{-bs}, r_\alpha - Ce^{-bs}].$$

Let us define the following sets (corresponding tail level sets):

$$I_s := \bigcup_{\alpha\in\mathcal{A}} [l_\alpha + Ce^{-bs}, r_\alpha - Ce^{-bs}] \text{ for any } s \geq \underline{g}.$$

Then $\xi(s) = dCe^{-bs}$ and $\Xi(s) = (dC/b)e^{-bs}$, so they satisfy the assumptions of Lemma 7.4. In view of Lemma 7.4, taking $c(s) = \frac{a}{b}\log s$ for some $a > 1$ we have

$$Leb(I^g \setminus A^s_{c(s)}) \leq sdCs^{-a} + 2\frac{a}{b}(\log s)dCs^{-a} + \frac{dC}{b}s^{-a} = O(s^{-(a-1)}), \tag{7.11}$$

so the measure of $I^g \setminus A^s_{c(s)}$ decays with the polynomial speed.

7.1.4. *Estimates of integrals and tails.* We can now combine the results on the two previous subsections, i.e. the reduction of integrals along the flow to Birkhoff sums (Lemma 7.1) and the decomposition of Birkhoff sums into special Birkhoff sums (Propostion 7.3), to get the following estimate of ergodic integrals in terms of special Birkhoff sums:

Proposition 7.7. *Let $\eta : \mathbb{R}_{\geq 0} \to [\underline{g}, +\infty)$ be an increasing C^1-map. Let $f : I^g \to \mathbb{R}$ be a measurable bounded map. Then, for every $s \in \mathbb{R}_{\geq 0}$,*

$$Leb(I^g \setminus A^s_{\eta(s)}) \leq s\ \xi(\eta(s)) + 2\eta(s)\xi(\eta(s)) + \Xi(\eta(s)) \tag{7.12}$$

(where $\xi(\,\cdot\,)$ is defined by (7.8) and $\Xi(\,\cdot\,)$ is given by Lemma 7.4), and for every $(x, r) \in A^s_{\eta(s)}$,

$$\Big| \int_0^s f(T^g_t(x, r))dt \Big| \leq 2\sum_{k\geq 0} \|Z(k+1)\| \|S(k)\varphi_f\|_{L^\infty(I^{(k)}(\eta(s)))} + 2\|f\|_{L^\infty}\eta^2(s) \tag{7.13}$$

Proof. The result follows by combining Lemma 7.1, Proposition 7.3 and Lemma 7.4 with $c = \eta(s)$. □

7.2. **Deviation spectrum and asymptotic behaviour of ergodic integrals.** We present in this section the proof of Theorem 1.4 and the first part of Theorem 1.3, namely the existence of the asymptotic spectrum for ergodic integrals both in the minimal and non-minimal case. We first define, in § 7.2.1, the cocycles that will govern the asymptotic behaviour of the ergodic integrals. Notice that, since we are proving at the same time the existence of the expansions in Theorems 1.3 and 1.4, we will define cocycles u_σ parametrized by $\sigma \in Fix(\psi_{\mathbb{R}}) \cap M'$ also when considering the restriction of a typical $\psi_{\mathbb{R}} \in \mathcal{U}_{\neg min}$ to a minimal component M' (even if these do not appear explicitely in the statement of Theorem 1.4, where they are absorbed in $err(f, T, \cdot)$). We then estimate the *error term* and shows that it exhibit subpolynomial deviations, see § 7.2.2 and then prove in § 7.2.3 that the cocycles that we build have the desired *pure power* behaviour, i.e. each has oscillations of the order of T^{ν_i} where ν_i is one of the g distinct exponents in the power spectrum. Finally, in § 7.2.4 we conclude the proof.

7.2.1. *Definition of the distributions and the cocycles.* Assume that $T = T_{(\pi,\lambda)}$ satisfies the UDC. Then, in view of the Oseledets genericity property (O) of the UDC condition (refer to Definition 3) there exists vectors $h_1, \ldots, h_g \in H(\pi^{(0)})$ such that

$$\lim_{k\to+\infty} \frac{1}{k}\|Q(k)h_i\| = \lambda_i \text{ for } 1 \leq i \leq g. \tag{7.14}$$

and furthermore $\operatorname{span}\{h_1, \ldots, h_g\} \oplus \Gamma_s^{(0)} = H(\pi^{(0)})$. We will now use these vectors h_i to define the distributions and the cocycles which appear in the asymptotic expansion.

The distributions. By Theorem 6.1 (in view of Remark 6.3) and Corollary 6.2 applied to $F := \operatorname{span}\{h_1, \ldots, h_g\}$, there exists a bounded operator $\mathfrak{h} : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to F$, such that $\mathfrak{h}(h) = h$ for every $h \in F$ and for every $\tau > 0$ if $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\mathfrak{h}(\varphi) = 0$ then

$$\frac{\|S(k)\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} = O(e^{\tau k}). \tag{7.15}$$

Let $d_i : \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha) \to \mathbb{R}$, $i = 1, \ldots, g$ be bounded operators such that

$$\mathfrak{h}(\varphi) = \sum_{i=1}^{g} d_i(\varphi) h_i \text{ for every } \varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha). \tag{7.16}$$

We can then define bounded operators $D_i : C^{2+\epsilon}(M) \to \mathbb{R}$, for $i = 1, \ldots, g$, by using the map $f \mapsto \varphi_f$ (see Proposition 4.1 for its basic properties) which associates to an observable $f : M \to \mathbb{R}$ the cocycle which arise in the skew-product representation of the Poincaré map described in § 2.3.3 and setting

$$D_i(f) := d_i(\varphi_f), \qquad 1 \leq i \leq g. \tag{7.17}$$

We will prove in § 7.2.4 that these are the distributions which enter in the asymptotic expansion.

The power growth cocycles. To construct the cocycles we exploit the following Lemma, proved in [10].

Lemma 7.8 (Lemma 7.4 in [10])**.** *For every $h \in H(\pi)$ there exists a $\mathcal{C}^\infty$-function $f : M \to \mathbb{R}$, which vanishes on a neighborhood* $\mathrm{Fix}(\psi_{\mathbb{R}})$*, such that $\varphi_f = h$.*

Let $f_i \in C^\infty(M)$ be the observable such that $\varphi_{f_i} = h_i$, given by Lemma 7.8 applied to $h = h_i$. Let us now define

$$u_i(T, x) := \int_0^T f_i(\psi_s(x))\, ds, \qquad \text{for } 1 \leq i \leq g.$$

The singular cocycles. For every $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'$, to define u_σ, let $\bar{\xi}_\sigma : M \to \mathbb{R}$ be any C^∞-map which is equal to 1 on an open neighbourhood of σ and equal to zero on an open neighbourhood of all other fixed points. Let $\xi_\sigma : M \to \mathbb{R}$ be a C^∞-map given by

$$\xi_\sigma := \bar{\xi}_\sigma - \sum_{i=1}^{g} D_i(\bar{\xi}_\sigma) f_i.$$

Then, since each f_i given by Lemma 7.8 vanishes on a neighbourhood of $Fix(\varphi_{\mathbb{R}})$ (see Lemma 7.8), ξ_σ is also equal to 1 on an open neighbourhood of σ and equal to zero on an open neighbourhood of all other fixed points. Moreover, by linearity of the operator $\mathfrak{h}$, the definition (7.17) of D_i and (7.16),

$$\mathfrak{h}(\varphi_{\xi_\sigma}) = \mathfrak{h}(\varphi_{\bar{\xi}_\sigma}) - \sum_{i=1}^{g} D_i(\bar{\xi}_\sigma)\mathfrak{h}(\varphi_{f_i}) = \mathfrak{h}(\varphi_{\bar{\xi}_\sigma}) - \sum_{i=1}^{g} d_i(\varphi_{\bar{\xi}_\sigma}) h_i = 0. \tag{7.18}$$

Finally, the cocycle $u_\sigma : \mathbb{R} \times M \to \mathbb{R}$ is defined by

$$u_\sigma(T, x) := \int_0^T \xi_\sigma(\psi_s(x))\, ds.$$

We will show in § 7.2.3 that each u_i, in view of (7.14), displays the desired deviation behaviour and in § 7.2.4 that they are indeed the desired asymptotic cocycles. We first estimate the error term though.

7.2.2. *Subpolynomial deviation case.* The following Proposition provides subpolynomial estimates for the growth of *corrected* ergodic integrals (in light of Corollary 6.2) and will be used in § 7.2.4 to control the *error term* in the asymptotic expansion.

Proposition 7.9 (Subpolynomial deviation). *Suppose that the IET $T: I \to I$ satisfies the UDC. Assume that $g, \varphi_f \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and*

$$\frac{1}{|I^{(k)}|}\|S(k)\varphi_f\|_{L^1(I^{(k)})} = O(e^{\tau k}) \text{ for every } \tau > 0.$$

Then for a.e. $(x,r) \in I^g$ we have

$$\limsup_{s\to+\infty} \frac{\log|\int_0^s f(T_t^g(x,r))dt|}{\log s} \leq 0. \tag{7.19}$$

Moreover, for every $p \geq 1$ we have

$$\limsup_{s\to+\infty} \frac{\log\|\int_0^s f\circ T_t^g dt\|_{L^p(I^g)}}{\log s} \leq 0. \tag{7.20}$$

Proof. As we have already seen in Remark 7.6, there exist two positive constants $C, b > 0$ such that for every $s \geq \underline{g}$ we have

$$g(x) \leq s \text{ for all } x \in I_s := \bigcup_{\alpha\in\mathcal{A}} [l_\alpha + Ce^{-bs}, r_\alpha - Ce^{-bs}].$$

Then

$$\xi(s) = Leb(I \setminus I_s) = dCe^{-bs} \quad \text{and} \quad \Xi(s) = (dC/b)e^{-bs}.$$

Take any $a > 2$ and set $\eta(s) = \frac{a}{b}\log s$. By the description of $C, b > 0$, we have $[0, Ce^{-b\eta(s)}] \subset I \setminus I_{\eta(s)}$. Hence, if $|I^{(k)}| \leq Ce^{-b\eta(s)} = C/s^a$ then $I^{(k)}(\eta(s)) = \emptyset$. By condition (UDC3) and (3.14), it follows that

$$I^{(k)}(\eta(s)) \neq \emptyset \Rightarrow |I^{(k)}| > C/s^a \Rightarrow \|Q(k)\| < \kappa s^a/C \Rightarrow k \leq \frac{a}{\lambda_1}\log(C's).$$

Moreover, if $x \in I^{(k)}(\eta(s)) \cap I_\alpha^{(k)}$ then

$$x \in [l_\alpha^{(k)} + Ce^{-b\eta(s)}, r_\alpha^{(k)} - Ce^{-b\eta(s)}] = [l_\alpha^{(k)} + C/s^a, r_\alpha^{(k)} - C/s^a].$$

In view of (4.12), (5.15) and (UDC3), it follows that for every $x \in I^{(k)}(\eta(s))$,

$$\begin{aligned}|(S(k)\varphi)(x)| &\leq 2\kappa\frac{\|S(k)\varphi\|_{L^1(I^{(k)})}}{|I^{(k)}|} + \mathcal{LV}(S(k)\varphi)(1+\log(|I^{(k)}|s^a/C))\\ &= O(e^{\tau k}) + O(\log s \log\|Q(k)\|) = O(e^{\tau k}) + O(k\log s).\end{aligned}$$

Therefore, by (7.13), for every $(x,r) \in A^s_{\eta(s)}$ we have

$$\begin{aligned}\Big|\int_0^s f(T_t^g(x,r))dt\Big| &\leq O(\log^2 s) + O\Big(\sum_{0\leq k\leq \frac{a}{\lambda_1}\log(C's)} \|Z(k+1)\|e^{\tau k}\Big) + O\Big(\log s \sum_{0\leq k\leq \frac{a}{\lambda_1}\log(C's)} \|Z(k+1)\|k\Big)\\ &\leq O(\log^2 s) + O(s^{2a\tau/\lambda_1}) + O(s^{a\tau/\lambda_1}\log^2 s) = O(s^{2a\tau/\lambda_1}).\end{aligned}$$

Moreover, by (7.11), we have $Leb(I^g \setminus A^s_{\eta(s)}) = O(1/s^{a-1})$ with $a - 1 > 1$. Therefore, for every $\tau > 0$ and $a > 2$ there exists $C_{\tau,a} > 0$ such that for every $s > 0$ we have

$$Leb\Big\{(x,r) \in I^g : \Big|\int_0^s f(T_t^g(x,r))dt\Big| > C_{\tau,a}s^{2\tau a/\lambda_1}\Big\} \leq Leb(I^g \setminus A^s_{\eta(s)}) < \frac{C_{\tau,a}}{s^{a-1}}. \tag{7.21}$$

It follows that for a.e. $(x,r) \in I^g$ we have

$$\limsup_{s\to+\infty} \frac{\log|\int_0^s f(T_t^g(x,r))dt|}{\log s} \leq 2\tau a/\lambda_1.$$

This gives (7.19).

Finally, the inequality (7.20) follows also directly from (7.21). Indeed, if $a \geq p+1$, then

$$\begin{aligned}\Big\|\int_0^s f\circ T_t^g dt\Big\|^p_{L^p(I^g)} &\leq \int_{A^s_{\eta(s)}} \Big|\int_0^s f\circ T_t^g(x,r)dt\Big|^p dx\,dr + Leb(I^g \setminus A^s_{\eta(s)})s^p\|f\|^p_{L^\infty}\\ &= O(s^{2pa\tau/\lambda_1}) + O(s^{p+1-a}) = O(s^{2pa\tau/\lambda_1}).\end{aligned}$$

□

Corollary 7.10. *Suppose that T is an IET satisfying the UDC and $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. If $\mathfrak{h}(\varphi) = 0$ then $\int_I \varphi(x)\,dx = 0$.*

Proof. Let us consider any roof function $g : I \to \mathbb{R}_{>0}$ such that $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $|\varphi(x)| \le g(x)$ for $x \in I$. Let $f : I^g \to \mathbb{R}$ be given by $f(x,r) = \varphi(x)/g(x)$ for $(x,r) \in I^g$. Then f is bounded and $\varphi_f = \varphi$. In view of Theorem 7.9 and the ergodicity of T, for every $0 < \tau < 1$, for a.e. $x \in I$ and a.e. $r \in [0, \underline{g}]$ we have

$$g^{(n)}(x) = O(n) \quad \text{and} \quad \int_0^{g^{(n)}(x)} f(T_t^g(x,r))\,dt = O((g^{(n)}(x))^\tau).$$

As

$$\begin{aligned}\Big|\varphi^{(n)}(x) - \int_0^{g^{(n)}(x)} f(T_t^g(x,r))\,dt\Big| &= \Big|\int_0^{g^{(n)}(x)} f(T_t^g(x,0))\,dt - \int_0^{g^{(n)}(x)} f(T_t^g(x,r))\,dt\Big| \\ &\le \Big|\int_0^r f(T_t^g(x,0))\,dt\Big| + \Big|\int_0^r f(T_t^g(T^n x,0))\,dt\Big| \le 2\underline{g}\|f\|_{\sup},\end{aligned}$$

it follows that $\varphi^{(n)}(x) = O(n^\tau)$. On the other hand, for a.e. $x \in I$ we have $\varphi^{(n)}(x)/n \to \int_I \varphi(x)\,dx$. This gives $\int_I \varphi(x)\,dx = 0$. □

7.2.3. *Pure power deviation case.* We consider first a function f such that $\varphi_f = h$, where h has exponential growth rate λ.

Proposition 7.11 (Pure deviation)**.** *Suppose that the IET $T : I \to I$ satisfies the UDC. Assume that the roof function $g \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $f : I^g \to \mathbb{R}$ is a bounded function such that there exists $K > 0$ for which $f(x,r) = 0$ for $r \ge K$ and $\varphi_f \in L^\infty(I)$. Suppose that for some $\lambda \ge 0$ we have*

$$\limsup_{k\to+\infty} \frac{\log \|S(k)(\varphi_f)\|_{L^\infty(I^{(k)})}}{k} \le \lambda.$$

Then

$$\limsup_{s\to+\infty} \frac{\log \|\int_0^s f\circ T_t^g\,dt\|_{L^\infty}}{\log s} \le \frac{\lambda}{\lambda_1}. \tag{7.22}$$

If additionally $\varphi_f = h = (h_\alpha)_{\alpha\in\mathcal{A}} \in H(\pi)$, $\lambda > 0$ and

$$\lim_{k\to+\infty} \frac{\log \|Q(k)h\|}{k} = \lambda,$$

then

$$\limsup_{s\to+\infty} \frac{\log \|\int_0^s f\circ T_t^g\,dt\|_{L^\infty}}{\log s} = \frac{\lambda}{\lambda_1}. \tag{7.23}$$

Proof. Let us consider the trimmed roof function $g_K : I \to [0,K]$, $g_K(x) = \min\{g(x), K\}$. Taking $\eta = K$ and $I_{\eta(s)} = I_K = I$ we have $A^s_{\eta(s)} = I^{g_K}$. Note that, by assumption, the map φ_f does not change after passing to the trimmed roof function. In view of (7.13), for every regular point $(x,r) \in I^{g_K}$ we have

$$\Big|\int_0^s f(T_t^{g_K}(x,r))dt\Big| \le 2\sum_{k=0}^{m(n_K(x,r,s))} \|Z(k+1)\|\|S(k)\varphi_f\|_{L^\infty(I^{(k)})} + 2\|f\|_{L^\infty}K^2,$$

where $n_K(x,r,s)$ is defined by (7.1) for the roof g_K. Then

$$0 \le n_K(x,r,s) \le n(x,r,s) \le s/\underline{g} + 1.$$

By assumption, for every $\tau > 0$ we have

$$\|S(k)\varphi_f\|_{L^\infty(I^{(k)})} = O(e^{(\lambda+\tau)k}).$$

Moreover, by (7.4),

$$e^{\lambda_1 m(n_K(x,r,s))} = O(n_K(x,r,s)^{1+\tau}) = O(s^{1+\tau}).$$

Therefore, by (3.16), it follows that

$$\begin{aligned}\Big|\int_0^s f(T_t^{g_K}(x,r))dt\Big| &\le O\Big(\sum_{k=0}^{m(n_K(x,r,s))} \|Z(k+1)\|\|S(k)\varphi_f\|_{L^\infty(I^{(k)})} + \|f\|_{L^\infty}K^2\Big) \\ &= O\Big(\sum_{k=0}^{m(n_K(x,r,s))} e^{(\lambda+2\tau)k} + \|f\|_{L^\infty}K^2\Big) \\ &= O\big(e^{(\lambda+2\tau)m(n_K(x,r,s))} + \|f\|_{L^\infty}K^2\big) \\ &= O(s^{(\lambda+2\tau)(1+\tau)}).\end{aligned}$$

By assumption and the definition of g_K, for every regular $(x,r)\in I^g$ and $s>0$ there exists $0\le s'=s'(x,r,s)\le s$ such that

$$\Big|\int_0^s f(T_t^g(x,r))dt\Big| = \Big|\int_0^{s'} f(T_t^{g_K}(x,r))dt\Big| = O\big((s')^{(\lambda+2\tau)(1+\tau)}\big) = O(s^{(\lambda+2\tau)(1+\tau)}).$$

This gives (7.22) and proves one inequality (namely the upper bound) in (7.23).

To prove the inverse inequality and therefore (7.23), note that for every $x\in I_\alpha^{(k)}$ we have

$$\int_0^{S(k)g(x)} f(T_t^g(x,0))\,dt = \int_0^{g^{(Q_\alpha(k))}(x)} f(T_t^g(x,0))\,dt = \varphi_f^{(Q_\alpha(k))}(x) = S(k)\varphi_f(x) = (Q(k)h)_\alpha.$$

Moreover, by assumption, for every $\tau>0$ there exists $c>0$ such that for every $k\ge 0$ we have

$$\sum_{\alpha\in\mathcal{A}} |(Q(k)h)_\alpha| = \|Q(k)h\| \ge ce^{\lambda(1-\tau)k}.$$

As g is positive, by (B1), (3.4) and (UDC3), we have

$$m(S(k)g, I_\alpha^{(k)}) \le \frac{|I^{(k)}|}{|I_\alpha^{(k)}|} m(S(k)g, I^{(k)}) \le \frac{\kappa m(g,I)}{|I^{(k)}|} \le \kappa m(g,I)\|Q(k)\| \le \kappa m(g,I)Ce^{\lambda_1(1+\tau)k}.$$

For every $k\ge 0$ choose $\alpha\in\mathcal{A}$ such that $|(Q(k)h)_\alpha| = \frac{1}{d}\|Q(k)h\|$ and then we take any $x^{(k)}\in I_\alpha^{(k)}$ such that $s_k := S(k)g(x^{(k)}) \le \kappa m(g,I)Ce^{\lambda_1(1+\tau)k}$. Then

$$\begin{aligned}\Big\|\int_0^{s_k} f\circ T_t^g\,dt\Big\|_{L^\infty} &\ge \Big|\int_0^{S(k)g(x^{(k)})} f(T_t^g(x^{(k)},0))\,dt\Big| = |(Q(k)h)_\alpha| \\ &= \frac{1}{d}\|Q(k)h\| \ge \frac{c}{d}e^{\lambda(1-\tau)k} \ge \frac{c}{d(\kappa m(g,I)C)^{\frac{\lambda}{\lambda_1}\frac{1-\tau}{1+\tau}}}(s_k)^{\frac{\lambda}{\lambda_1}\frac{1-\tau}{1+\tau}k}.\end{aligned}$$

It follows that for every $\tau>0$ we have

$$\limsup_{s\to+\infty}\frac{\log\|\int_0^s f\circ T_t^g\,dt\|_{L^\infty}}{\log s} \ge \frac{\lambda}{\lambda_1}\frac{1-\tau}{1+\tau},$$

which gives (7.23). □

To have uniform control over the asymptotics of the error growth, we also need the following Corollary.

Corollary 7.12. *Let $T: I\to I$ is an IET satisfying the UDC and $g\in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ be a roof function. Suppose that $f: I^g\to\mathbb{R}$ is a bounded function such that $\varphi_f, \varphi_{|f|}\in L^\infty(I)$. Then for every $\lambda\ge 0$,*

$$\limsup_{k\to+\infty}\frac{\log\|S(k)(\varphi_f)\|_{L^\infty(I^{(k)})}}{k} \le \lambda \implies \limsup_{s\to+\infty}\frac{\log\|\int_0^s f\circ T_t^g\,dt\|_{L^\infty}}{\log s} \le \frac{\lambda}{\lambda_1}. \tag{7.24}$$

Proof. For any $K>0$ let us consider the bounded map $f_K: I^g\to\mathbb{R}$ given by

$$f_K(x,r) = \begin{cases} f(x,r) & \text{if } g(x)\le K\\ \varphi_f(x)/K & \text{if } g(x)>K \text{ and } r\le g(x)\\ 0 & \text{if } g(x)>K \text{ and } r>g(x).\end{cases}$$

Then f_K satisfies the assumptions of the first part of Proposition 7.11 and $\varphi_{f_K}=\varphi_f$. Hence

$$\limsup_{s\to+\infty}\frac{\log\|\int_0^s f_K\circ T_t^g\,dt\|_{L^\infty}}{\log s} \le \frac{\lambda}{\lambda_1}. \tag{7.25}$$

Note that for every $x\in I$ in the interior of exchanged intervals and any pair $0\le r_1<r_2\le g(x)$ we have

$$|\int_{r_1}^{r_2} f(x,r)\,dr| \le \int_{r_1}^{r_2}|f(x,r)|\,dr \le \int_0^{g(x)}|f(x,r)|\,dr = \varphi_{|f|}(x) \le \|\varphi_{|f|}\|_{\sup},$$
$$|\int_{r_1}^{r_2} f_K(x,r)\,dr| \le \varphi_{|f_K|}(x)| \le \varphi_{|f|}(x) \le \|\varphi_{|f|}\|_{\sup}.$$

As

$$\int_0^{g(x)} f_K(x,r)\,dr = \varphi_{f_K}(x) = \varphi_f(x) = \int_0^{g(x)} f(x,r)\,dr,$$

it follows that for every regular point $(x,r)\in I^g$ and any $s>0$ we have

$$\Big|\int_0^s f(T_t^g(x,r))\,dt - \int_0^s f_K(T_t^g(x,r))\,dt\Big| \le 4\|\varphi_{|f|}\|_{\sup}.$$

Together with (7.25) this yields (7.24). □

7.2.4. *Power deviation spectrum.* Combining the results in the two previous subsections, we can now prove the full deviation spectrum result stated in Theorem 1.4 as well as the existence of the asymptotic expansion in Theorem 1.3.

Proof of Theorem 1.4 and of the first part of Theorem 1.3. Let $D_i, 1 \le i \le g$, $u_i, 1 \le i \le g$ and u_σ, $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'$, be respectively the distributions and the cocycles defined in § 7.2.1. One can see that, for each $1 \le i \le g$, u_i displays the desired power behaviour (1.6), by the pure deviation Theorem 7.11 proved in § 7.2.3, which can be applied to $f = f_i$ since by construction $\varphi_{f_i} = h_i$ and h_i has exponential growth rate λ_i, see (7.14).

The error term function. Let us consider $f_e \in C^{2+\epsilon}(M)$ given by

$$f_e := f - \sum_{i=1}^{g} D_i(f) f_i.$$

By the definition of f_i, $i = 1, \ldots, g$,

$$f_e(\sigma) = f(\sigma) \text{ for every } \sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'. \tag{7.26}$$

Then we set

$$err(f, T, x) := \int_0^T f_e(\psi_s(x))\, ds.$$

Let φ_{f_e} be the cocycle associated to f_e (refer to § 2.3.3). We can then check that $\mathfrak{h}(\varphi_{f_e}) = 0$, since

$$\mathfrak{h}(\varphi_{f_e}) = \mathfrak{h}(\varphi_f) - \sum_{i=1}^{g} D_i(f)\mathfrak{h}(\varphi_{f_i}) = \mathfrak{h}(\varphi_f) - \sum_{i=1}^{g} D_i(f)\mathfrak{h}(h_i) = \mathfrak{h}(\varphi_f) - \sum_{i=1}^{g} d_i(\varphi_f) h_i = 0. \tag{7.27}$$

We now show that for every non-zero $\xi \in C^{2+\epsilon}(M)$ such that $\mathfrak{h}(\varphi_\xi) = 0$ we have

$$\limsup_{T\to+\infty} \frac{\log |\int_0^T \xi(\psi_t(x))\, dt|}{\log T} = 0 \text{ for a.e. } x \in M', \quad \limsup_{T\to+\infty} \frac{\log \|\int_0^T \xi \circ \psi_t\, dt\|_{L^p(M')}}{\log T} = 0. \tag{7.28}$$

As $\mathfrak{h}(\varphi_\xi) = 0$, in view of Corollary 6.2, we can apply the subpolynomial deviation Theorem 7.9 to $f = \xi$ and prove both inequalities $\le$ in (7.28).

Almost everywhere error estimates. Suppose now that the left equality in (7.28) does not hold. Then there exists a subset $B \subset M'$ with positive area such that

$$\lim_{T\to+\infty} \int_0^T \xi(\psi_t(x))\, dt = 0 \text{ for all } x \in B.$$

By the ergodicity of the flow, for μ-a.e. $x \in M'$, the limit

$$\zeta(x) = \lim_{T\to+\infty} \int_0^T \xi(\psi_t(x))\, dt \text{ exists.}$$

Then $\zeta : M' \to \mathbb{R}$ is a measurable map such that $\zeta(x) = 0$ for any $x \in B$ and

$$\zeta(x) - \zeta(\psi_s x) = \int_0^s \xi(\psi_t(x))\, dt \text{ for every } s > 0 \text{ and a.e. } x \in M'. \tag{7.29}$$

Note that $\zeta \equiv 0$. Indeed, by definition and (7.29), for a.e. $x \in M'$ we have $\lim_{s\to+\infty} \zeta(\psi_s x) = 0$. Since $\psi_{\mathbb{R}}$ is ergodic, this gives $\zeta \equiv 0$. Therefore,

$$\frac{1}{s}\int_0^s \xi(\psi_t(x))\, dt = \frac{1}{s}(\zeta(x) - \zeta(\psi_s x)) = 0 \text{ for every } s > 0 \text{ and a.e. } x \in M'.$$

As ξ is continuous, it follows that for a.e. $x \in M'$ we have

$$\xi(x) = \lim_{s\to 0} \frac{1}{s} \int_0^s \xi(\psi_t(x))\, dt = 0,$$

contrary to the assumption ξ is non-zero.

Error estimates in L^p norm. Suppose now that the right equality in (7.28) does not hold. Then

$$\lim_{T\to+\infty} \int_0^T \xi \circ \psi_t\, dt = 0 \text{ in } L^p.$$

Hence, for every $s > 0$ we have

$$\int_0^s \xi \circ \psi_t\, dt = \lim_{T\to+\infty} \int_0^{T+s} \xi \circ \psi_t\, dt - \lim_{T\to+\infty} \int_0^T \xi \circ \psi_t \circ \psi_s\, dt = 0$$

in L^p. It follows that $\frac{1}{s}\int_0^s \xi(\psi_t(x))\, dt = 0$ for every $s > 0$ and a.e. $x \in M'$. The final contradiction argument is the same as above. This completes the proof of (7.28).

In view of (7.18) and (7.27), $\mathfrak{h}(\varphi_{\xi_\sigma}) = 0$ and $\mathfrak{h}(\varphi_{f_e}) = 0$, so we can apply (7.28) to $\xi = \xi_\sigma$ and $\xi = f_e$. This yields (1.7) in in Theorem 1.3 as well as (1.12) and (1.13) in Theorem 1.4.

Uniform estimates of err_b. Let us consider $f_{eb} \in C^{2+\epsilon}(M)$ given by

$$f_{eb} = f_e - \sum_{\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'} f(\sigma)\xi_\sigma. \tag{7.30}$$

Then

$$\begin{aligned}\int_0^T f_{eb}(\psi_t x)\, dt &= \int_0^T f_e(\psi_t x)\, dt - \sum_{\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'} f(\sigma) \int_0^T \xi_\sigma(\psi_t x)\, dt \\ &= err(f, T, x) - \sum_{\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'} f(\sigma) u_\sigma(T, x) = err_b(f, T, x).\end{aligned}$$

Since $f_e(\sigma) = f(\sigma)$ and $\xi_\sigma(\sigma') = \delta_{\sigma,\sigma'}$, for every $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'$ we have

$$f_{eb}(\sigma) = f_e(\sigma) - \sum_{\sigma' \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'} f(\sigma')\xi_{\sigma'}(\sigma) = 0. \tag{7.31}$$

In view of Proposition 4.1, $\varphi_{f_{be}} \in \mathrm{AC}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$. As $\mathfrak{h}(\varphi_{\xi_\sigma}) = 0$ and $\mathfrak{h}(\varphi_{f_e}) = 0$, we also have

$$\mathfrak{h}(\varphi_{f_{eb}}) = \mathfrak{h}(\varphi_{f_e}) - \sum_{\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}}) \cap M'} f(\sigma)\mathfrak{h}(\varphi_{\xi_\sigma}) = 0. \tag{7.32}$$

In view of (4.13), the property (B1) of the UDC, (5.3) and Corollary 6.2, it follows that for every $\tau > 0$ we have

$$\begin{aligned}\|S(k)\varphi_{f_{eb}}\|_{\sup} &\le \frac{|I^{(k)}|}{\min_{\alpha \in \mathcal{A}} |I_\alpha^{(k)}|} \frac{\|S(k)\varphi_{f_{eb}}\|_{L^1(I^{(k)})}}{|I^{(k)}|} + \mathrm{Var}(S(k)\varphi_{f_{eb}}) \\ &\le \kappa \frac{\|S(k)\varphi_{f_{eb}}\|_{L^1(I^{(k)})}}{|I^{(k)}|} + \mathrm{Var}(\varphi_{f_{eb}}) = O(e^{\tau k}).\end{aligned}$$

By (7.31) and Proposition 4.1 (see in particular property (i)), the map $\varphi_{|f|} : I \to \mathbb{R}$ is bounded. In view of Corollary 7.12, this gives (1.11).

This completes the proof of Theorem 1.4 as well as the proof of the first part of Theorem 1.3. □

The second part of the statement of Theorem 1.3, namely the equidistribution statement for the error term (which is a consequence of ergodicity) and the uniform estimates on err_b will be proved at the end, in § 8.2.4.

8. Ergodicity of extensions

The goal of this section is to prove Main Theorem 1.2 and complete the proof of Main Theorem 1.3. In view of the reduction explained in § 2.3.3 and the equivalence between ergodicity of the extension $\Phi^f_{\mathbb{R}}$ on $M \times \mathbb{R}$ and of the skew product T_{φ_f} on $I \times \mathbb{R}$ obtained via a Poincaré first return, we treat first the case of skew products of this form. The main result on ergodicity of skew products is Theorem 8.1 stated in § 8.1 below. In § 8.1.1 we state the ergodicity criterium that we will use to prove it (see Proposition 8.2). Theorem 8.1 is then proved in § 8.1.2. Finally, in § 8.2 we prove Main Theorem 1.2, by combining the ergodicity result for skew products with a discussion on reducibility.

8.1. **Ergodicity of skew products over IETs with logarithmic singularities.** We state in this section the ergodicity result for skew-products over IETs with cocycles with logarithmic singularities. We also show that the ergodicity result for locally Hamiltonian flows (Main Theorem 1.2) can be reduced to it.

Theorem 8.1 (Ergodicity of skew-products with log-singularities over IETs)**.** *Suppose that* $T : I \to I$ *satisfies the SUDC. Let* $\varphi \in \mathrm{LG}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha)$ *be a cocycle with logarithmic singularities of geometric type so that*

$$\mathcal{L}(\varphi) > 0, \qquad \mathcal{AS}(\varphi) = 0, \qquad g'_\varphi \in \mathrm{LG}(\sqcup_{\alpha \in \mathcal{A}} I_\alpha).$$

Assume furthermore that φ *is* corrected*, namely* $\mathfrak{h}(\varphi) = 0$*. Then the skew product* T_φ *on* $I \times \mathbb{R}$ *is ergodic.*

The proof of the Theorem will take most of the section, from § 8.1 to the end. We first state the ergodicity criterium which will be exploited (see § 8.1.1) and proceed with the proof, which will take § 8.1.2.

8.1.1. *An ergodicity criterium.* We now formulate a quite classical criterium (Proposition 8.2) for ergodicity of a special flow. It shows that one can deduce the existence of *essential values* (a classical tool to prove ergodicity, see e.g. [1, 59]) to the presence of *rigidity sets* for the base transformation on which Birkhoff sums (up to the time which gives rigidity) are *tight*. The criterium was in particular used (and proved) in [20]. For simplicity in this section we constantly assume that $|I| = 1$.

We first give the definition of *rigidity sequence* for IETs (which are the base transformations in the special flow).

Definition 9 (Rigidity sequences for an IET). Let $T : I \to I$ be an IET. Let $(\Xi_n)_{n\geq 1}$ be a sequence of towers of intervals of the form $\Xi_n = \{T^i J_n : 0 \leq i < p_n\}$. We say that $(\Xi_n)_{n\geq 1}$ is a *rigid sequence of towers* if there exists a strictly increasing sequence $(q_n)_{n\geq 1}$, called the *rigidity sequence*, and $\delta > 0$ such that

$$Leb(\Xi_n) \geq \delta \text{ and } \sup_{x\in\Xi_n} |T^{q_n}x - x| \to 0.$$

The following Proposition is the ergodicity criterium that we will exploit. It was proved in [20] (using Proposition 2.3 and the end of the proof of Proposition 5.2 in [20]).

Proposition 8.2 (Ergodicity criterium, see [20])**.** *Assume that $T : I \to I$ is an ergodic IET and $\varphi : I \to \mathbb{R}$ a measurable map. Suppose that $(\Xi_n)_{n\geq 1}$ is a rigid sequence of tower and $(q_n)_{n\geq 1}$ its rigidity sequence. If for all $|s| \geq s_0$ we have*

$$\int_{\Xi_n} |\varphi^{(q_n)}(x)|\, dx = O(1) \text{ and } \int_{\Xi_n} e^{2\pi s\varphi^{(q_n)}(x)}\, dx = \frac{2}{3}Leb(\Xi_n) + O(|s|^{-1}), \tag{8.1}$$

then the skew product T_φ on $I \times \mathbb{R}$ is ergodic.

Remark 8.3. Suppose that T satisfies the SUDC. For every $k \geq 1$ let $J^{(k)} \subset I^{(k)}_{\alpha_k}$ be a sequence of intervals such that $\liminf |J^{(k)}|/|I^{(k)}_{\alpha_k}| > 0$. Then $\Xi_k = \{T^i J^{(k)} : 0 \leq i < p_k\}$ establishes a rigid sequence of towers with the rigidity sequence given by $q_k := Q_{\alpha_k}(k)$. It follows directly from (B1) and (B2). Since $T^{q_k} J^{(k)} \subset I^{(k)}$, for every $0 \leq l \leq q_k$ we have that $T^l\Xi_k = \{T^{l+i}J^{(k)} : 0 \leq i < p_k\}$ is also a tower of intervals.

Specializing the ergodicity criterion to our setting, we have the following Proposition, that shows that to prove ergodicity (and Theorem 8.1) it is sufficient to verify the assumptions in the statement:

Proposition 8.4. *Suppose that T satisfies the SUDC and let $(\Xi_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ be a sequence of rigid towers and its rigidity sequence as in Remark 8.3. Let $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ be a map such that $g'_\varphi \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. We additionally assume that there exists $c > 0$ such that*

- *(i) the sequence $\frac{1}{|I^{(k)}|}\|S(k)\varphi\|_{L^1(I^{(k)})}$ is bounded;*
- *(ii) $\mathrm{dist}\left(\bigcup_{i=0}^{q_k} T^i\Xi_k, End(T)\right) \geq c/q_k$;*
- *(iii) for every $0 \leq j < p_k$ there exists an interval $J^{(k)}_j \subset T^j J^{(k)}$ such that $|J^{(k)}_j| \geq |J^{(k)}|/3$ and $|(\varphi')^{(q_k)}(x)| \geq cq_k$ for all $x \in J^{(k)}_j$.*

Then the skew product T_φ on $I \times \mathbb{R}$ is ergodic.

The proof is a variation on arguments from [20]. We present it for completeness in the Appendix A.1.

Remark 8.5. One can see from the proof presented in Appendix A.1 that the same conclusion about the ergodicity of the skew product can be deduced under the assumption that conditions (i), (ii) and (iii) hold along any subsequence.

8.1.2. *Proof of ergodicity of skew products.* We will now prove Theorem 8.1 by showing that the assumptions of the criterion for ergodicity of skew products with logarithmic singularities over IETs (namely Proposition 8.4) hold.

For every $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ such that $\mathcal{AS}(\varphi) = 0$ and $\mathcal{L}(\varphi) > 0$ (using the definitions introduced in § 4) we want to construct a sequence of rigid towers as in Remark 8.3 for which the condition (ii) and (iii) in Proposition 8.4 hold. In view of (B2) and [20, Lemma 5.1], we have the following result:

Lemma 8.6. *Let $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ be such that $\mathcal{AS}(\varphi) = 0$ and $\mathcal{L}(\varphi) > 0$. There exists a sequence $(\alpha_k)_{k\geq 1}$ in $\mathcal{A}$ and a sequence on natural number $(j_k)_{k\geq 1}$ such that $p_k \leq j_k < Q_{\alpha_k}(k)$ and at least one of the following cases hold:*

- *(L): $C^+_{\alpha_0} \neq 0$ and $T^{j_k} l^{(k)}_{\alpha_k} = l_{\alpha_0}$ or*
- *(R): $C^-_{\alpha_0} \neq 0$ and $\widehat{T}^{j_k} r^{(k)}_{\alpha_k} = r_{\alpha_0}$.*

Moreover, the closures of the intervals $T^j I^{(k)}_{\alpha_k}$ for $Q_{\alpha_k}(k) \leq j < Q_{\alpha_k}(k) + p_k$ do not intersect $End(T)$.

Definition 10. For any $0 \leq \bar{c} < 1/2$ we define the base $J^{(k)} \subset I^{(k)}_{\alpha_k} = [a,b)$ of the tower Ξ_k as follows:

$$J^{(k)} = \Big(a + \frac{\bar{c}}{2}\lambda^{(k)}_{\alpha_k}, a + \bar{c}\lambda^{(k)}_{\alpha_k}\Big) \text{ in case } (L);$$

$$J^{(k)} = \Big(a - \bar{c}\lambda^{(k)}_{\alpha_k}, b - \frac{\bar{c}}{2}\lambda^{(k)}_{\alpha_k}\Big) \text{ in case } (R).$$

Lemma 8.7. *Let $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ be such that $\mathcal{AS}(\varphi) = 0$ and $\mathcal{L}(\varphi) > 0$ and $g'_\varphi, g''_\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. Let $(\Xi_k)_{k\geq 1}$ be a sequence of rigid towers defined in Definition 10 with*

$$\bar{c} = \sqrt{\frac{|C^{\pm}_{\alpha_0}|}{2(6\mathcal{L}(\varphi) + \mathrm{Var}\, g'_\varphi)}}. \tag{8.2}$$

Then

$$|(\varphi'')^{(q_k)}(x)| \geq \frac{\underline{c}}{(\lambda^{(k)}_{\alpha_k})^2} \text{ for all } x \in \Xi_k,$$

where $\underline{c} = 6\mathcal{L}(\varphi) + \mathrm{Var}\, g'_\varphi$.

Proof. We present the proof only in the case (L). The other case is similar.

Suppose that $x \in T^l J^{(k)}$ for some $0 \leq l < p_k$. By assumption,

$$\frac{|C^+_{\alpha_0}|}{\{T^{j_k - l}x - l_{\alpha_0}\}^2} \geq \frac{|C^+_{\alpha_0}|}{\bar{c}^2(\lambda^{(k)}_{\alpha_k})^2},$$

the elements of the orbit $T^j x$ for $0 \leq j < p_k$ are distant from each other at least $\lambda^{(k)}_{\alpha_k}$ and for $j \neq j_k - l$ we have $\{T^j x - l_{\alpha_0}\} \geq \lambda^{(k)}_{\alpha_k}$. It follows that

$$\sum_{0\leq j<p_k,\ j\neq j_k - l} \frac{|C^+_{\alpha_0}|}{\{T^j x - l_{\alpha_0}\}^2} \leq \frac{|C^+_{\alpha_0}|}{(\lambda^{(k)}_{\alpha_k})^2}\sum_{j=1}^{p_k}\frac{1}{j^2} \leq \frac{\pi^2}{6}\frac{|C^+_{\alpha_0}|}{(\lambda^{(k)}_{\alpha_k})^2} \leq 2\frac{|C^+_{\alpha_0}|}{(\lambda^{(k)}_{\alpha_k})^2}.$$

Since for every $0 \leq j < p_k$ we have $\{T^j x - l_\alpha\} \geq \lambda^{(k)}_{\alpha_k}$ for $\alpha \neq \alpha_k$ and $\{r_\alpha - T^j x\} \geq \lambda^{(k)}_{\alpha_k}/2$ for all $\alpha \in \mathcal{A}$, the same arguments show that for all $\alpha \in \mathcal{A}$ we have

$$\sum_{0\leq j<p_k} \frac{|C^-_\alpha|}{\{r_\alpha - T^j x\}^2} \leq 6\frac{|C^-_\alpha|}{(\lambda^{(k)}_{\alpha_k})^2} \text{ and } \sum_{0\leq j<p_k} \frac{|C^+_\alpha|}{\{T^j x - l_\alpha\}^2} \leq 2\frac{|C^+_\alpha|}{(\lambda^{(k)}_{\alpha_k})^2} \text{ if } \alpha \neq \alpha_k.$$

Moreover, for every $x \in I$ we have

$$|(g''_\varphi)^{(q_k)}(x)| \leq q_k \|g''_\varphi\|_{\sup} \leq \frac{\mathrm{Var}\, g'_\varphi}{(\lambda^{(k)}_{\alpha_k})^2}.$$

As

$$\varphi''(x) = \sum_{\alpha\in\mathcal{A}} \frac{C^+_\alpha}{\{x - l_\alpha\}^2} + \sum_{\alpha\in\mathcal{A}} \frac{C^-_\alpha}{\{r_\alpha - x\}^2} + g''_\varphi(x),$$

it follows that for every $x \in \Xi_k$ we have

$$|(\varphi'')^{(q_k)}(x)| \geq \frac{|C^+_{\alpha_0}|}{\bar{c}^2(\lambda^{(k)}_{\alpha_k})^2} - \frac{6\mathcal{L}(\varphi)}{(\lambda^{(k)}_{\alpha_k})^2} - \frac{\mathrm{Var}\, g'_\varphi}{(\lambda^{(k)}_{\alpha_k})^2} = \Big(\frac{|C^+_{\alpha_0}|}{\bar{c}^2} - 6\mathcal{L}(\varphi) - \mathrm{Var}\, g'_\varphi\Big)\frac{1}{(\lambda^{(k)}_{\alpha_k})^2} = \frac{\underline{c}}{(\lambda^{(k)}_{\alpha_k})^2}.$$

□

The following elementary lemma will help us to choose the subintervals $J^{(k)}_l \subset T^l J^{(k)}$ satisfying condition (iii) in Proposition 8.4.

Lemma 8.8. *Let $f : I \to \mathbb{R}$ be a C^1 map defined on a closed interval I and such that $|f'(x)| \geq c > 0$ for all $x \in I$. Then there exists a closed subinterval $J \subset I$ such that $|J| \geq |I|/3$ and $|f(x)| \geq c|I|/6$ for all $x \in I$.*

Proof of Theorem 8.1. Recall that $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is a cocycle such that

$$\mathcal{L}(\varphi) > 0, \quad \mathcal{AS}(\varphi) = 0, \quad \mathfrak{h}(\varphi) = 0 \quad \text{and} \quad g'_\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha).$$

By definition, $\varphi = \varphi_0 + g_\varphi$, where

$$\varphi_0(x) = -\sum_{\alpha\in\mathcal{A}} C^+_\alpha \log\{x - l_\alpha\} - \sum_{\alpha\in\mathcal{A}} C^-_\alpha \log\{r_\alpha - x\}$$

and $g_\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $g'_\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. By Proposition 8.9, g_φ is cohomologous to a piecewise linear map $\psi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathfrak{h}(\psi) = \mathfrak{h}(g_\varphi)$. It follows that φ is cohomologous to $\bar{\varphi} := \varphi_0 + \psi$. Then $\bar{\varphi} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is such that $\mathcal{L}(\bar{\varphi}) = \mathcal{L}(\varphi_0) = \mathcal{L}(\varphi) > 0$, $\mathcal{AS}(\bar{\varphi}) = \mathcal{AS}(\varphi_0) = \mathcal{AS}(\varphi) = 0$,

$$\mathfrak{h}(\bar{\varphi}) = \mathfrak{h}(\varphi_0) + \mathfrak{h}(\psi) = \mathfrak{h}(\varphi_0) + \mathfrak{h}(g_\varphi) = \mathfrak{h}(\varphi) = 0$$

and $g_{\bar{\varphi}} = \psi$, so $g'_{\bar{\varphi}}, g''_{\bar{\varphi}} \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. As φ is cohomologous to $\bar{\varphi}$, the skew products T_φ and $T_{\bar{\varphi}}$ are isomorphic, so it is sufficient to show the ergodicity of $T_{\bar{\varphi}}$.

Let $(\Xi_k)_{k\geq 1}$ be a sequence of rigid towers defined in Definition 10 with $\bar{c}$ given by (8.2) for that function $\bar{\varphi}$. In view of (B2) and (3.15), this sequence satisfies

$$\mathrm{dist}\Big(\bigcup_{i=0}^{q_k} T^i\Xi_k, End(T)\Big) \geq \frac{1}{2}\bar{c}\lambda^{(k)}_{\alpha_k} \geq \frac{\delta\bar{c}}{2\kappa q_k},$$

so condition (ii) in Proposition 8.4 holds. Moreover, by Lemma 8.7,

$$|(\bar{\varphi}'')^{(q_n)}(x)| \geq \frac{\underline{c}}{(\lambda^{(k)}_{\alpha_k})^2} \text{ for every } x \in T^l J^{(k)},\ 0 \leq l < p_k.$$

Since $|T^l J^{(k)}| = |J^{(k)}| = \frac{\bar{c}}{2}\lambda^{(k)}_{\alpha_k}$, by Lemma 8.8, for every $0 \leq l < p_k$ there exists an interval $J^{(k)}_l \subset T^l J^{(k)}$ such that

$$|J^{(k)}_l| \geq |J^{(k)}|/3 \text{ and } |(\bar{\varphi}')^{(q_n)}(x)| \geq \frac{\underline{c}}{(\lambda^{(k)}_{\alpha_k})^2}\frac{\bar{c}}{12}\lambda^{(k)}_{\alpha_k} \geq \frac{\underline{c}\bar{c}}{12}q_k \text{ for every } x \in J^{(k)}_l,$$

so condition (iii) in Proposition 8.4 holds.

Since $\mathcal{AS}(\bar{\varphi}) = 0$ and $\mathfrak{h}(\bar{\varphi}) = 0$, by Theorem 6.1 and (3.26) in Proposition 3.9,

$$\frac{\|S(r_n)\bar{\varphi}\|_{L^1(I^{(r_n)})}}{|I^{(r_n)}|} \text{ is bounded,}$$

so condition (i) in Proposition 8.4 holds along a subsequence. In view of Proposition 8.4 together with Remark 8.5, this gives the ergodicity of $T_{\bar{\varphi}}$, and hence the ergodicity of T_φ. □

8.2. **Reducibility and final arguments.** The main goal of this section is to prove Main Theorem 1.2, in particular the dichotomy between ergodicity and reducibility for typical extensions with observables in a suitable subspace of smooth functions. We also deduce from Main Theorem 1.2 the second and final part of Main Theorem 1.3. We first need to state an auxiliary result that we call *cohomological reduction*.

8.2.1. *Cohomological reduction.* The following result allows to reduce the study of cocycles whose derivatives have logarithmic singularities (up to coboundaries and hence cohomological equivalence) to piecewise linear cocycles (whose derivative is piecewise-constant). An analogous result was proved also in [20], but only in the special measure zero class of self-similar IETs considered there.

Theorem 8.9. *Assume that T satisfies the UDC. Then every $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\varphi' \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is cohomologous (via a bounded transfer function) to a piecewise linear cocycle $\psi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\mathfrak{h}(\psi) = \mathfrak{h}(\varphi)$, $\partial_\pi(\psi) = \partial_\pi(\varphi)$ and $\|S(k)(\varphi - \psi)\|_{\sup}$ tends to 0 exponentially.*

The proof of the theorem, which generalizes the proof in [20] to full measure, is included in the Appendix A.2. In view of Theorem A in [44], if T is a Roth-type IET and $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is such that $s(\varphi) = 0$ and $\varphi' \in \mathrm{BV}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, then φ is cohomologous (via a bounded transfer function) to a piecewise constant map h and $\|S(k)(\varphi - h)\|_{\sup}$ tends to 0 exponentially.

Assume additionally that φ in Theorem 8.9 satisfies $s(\varphi) = 0$. Then

$$s(\psi) = \sum_{\mathcal{O}\in\Sigma(\pi)} (\partial_\pi(\psi))_{\mathcal{O}} = \sum_{\mathcal{O}\in\Sigma(\pi)} (\partial_\pi(\varphi))_{\mathcal{O}} = s(\varphi) = 0.$$

So, by Theorem A in [44], it follows that ψ is cohomologous to piecewise constant map. As the UDC implies Roth-type (see Remark 3.7), this gives the following important corollary, which gives a generalization of Theorem A in [44].

Corollary 8.10. *Assume that T satisfies the UDC. Then every $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $s(\varphi) = 0$ and $\varphi' \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is cohomologous (via a bounded transfer function) to a piecewise constant map h and $\|S(k)(\varphi - h)\|_{\sup}$ tends to 0 exponentially.*

The importance of this result is that in view of Proposition 4.1 it applies to solve cohomological equations for a.e. $\psi_{\mathbb{R}} \in \mathscr{U}_{\min}$ and for functions $f \in C^{2+\epsilon}(M)$ vanishing on $\mathrm{Fix}(\psi_{\mathbb{R}})$. Recall that Theorem A in [44] applies only when f vanishes on an open neighborhood of $\mathrm{Fix}(\psi_{\mathbb{R}})$.

Classical Gottschalk-Hedlund type arguments, first applied in the context of IETs in [44, §3.4], show the following.

Lemma 8.11 ([44])**.** *Suppose that $T : I \to I$ is a minimal IET and $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. The following conditions are equivalent:*

(i) the sequence $\|\varphi^{(n)}\|_{\sup}$, $n \in \mathbb{N}$, is bounded;
(ii) $\varphi = g - g \circ T$, where $g : I \to \mathbb{R}$ is bounded;
(iii) $\varphi = g - g \circ T$, where $g : I \to \mathbb{R}$ is bounded and has at most countably many discontinuities.

Proof. The implications follow from the classical Gottschalk-Hedlund theorem, that can be applied to IETs by extending them to a homeomorphism to a Cantor space, see [44, §3.4]. The only non-classical implication, (iii)⇒(i) is also proved in [44, §3.4], where the authors show that the transfer map g exists and is the composition of a continuous map and a monotonic map, so is bounded and has at most countably many discontinuities. □

8.2.2. *Reduction to coboundaries.* Now that we reduced to the study of cocycles which are piecewise absolutely continuous (i.e. to $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$), we can prove reducibility exploiting the following result on coboundaries.

Proposition 8.12. *Assume that T satisfies the UDC. Then every $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\partial_\pi(\varphi) = 0$, $\mathfrak{h}(\varphi) = 0$ and $\varphi' \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ is a coboundary with a bounded transfer map having at most countably many discontinuities.*

Proof. By Corollary 8.10, there exists $h \in \Gamma$ and a bounded map $g : I \to \mathbb{R}$ such that $\varphi = h + g - g \circ T$. Moreover, $\|S(k)(\varphi - h)\|_{\sup}$ tends to 0 exponentially. As

$$\|\partial_\pi(\varphi - h)\| = \|\partial_{\pi^{(k)}}(S(k)(\varphi - h))\| \leq 2d\|S(k)(\varphi - h)\|_{\sup},$$

it follows that $\partial_\pi(h) = \partial_\pi(\varphi) = 0$, so $h \in H(\pi)$. Moreover, as

$$\frac{\|S(k)(\varphi - h)\|_{L^1(I^{(k)})}}{|I^{(k)}|} \leq \|S(k)(\varphi - h)\|_{\sup} \to 0,$$

by the definition of the operator $\mathfrak{h}$ and Corollary 6.2, we have $\mathfrak{h}(\varphi - h) = 0$. It follows that $\mathfrak{h}(h) = \mathfrak{h}(\varphi) = 0$, so $h \in \Gamma_s$. Therefore, h is also a coboundary with a bounded transfer map. As the sum of two coboundaries, $\varphi = h + g - g \circ T$ is also a coboundary with a bounded transfer map. Finally, in view of Lemma 8.11, the transfer map has at most countably many discontinuities. □

8.2.3. *Proof of the dichotomy for extensions.* We have now all ingredients needed for the proof of the dichotomy in Main Theorem 1.2.

Proof of Main Theorem 1.2. Let us say that a locally Hamiltonian flows $\psi_{\mathbb{R}}$ satisfies the SUDC condition if and only if $\psi_{\mathbb{R}}$ has a section $I \subset M$ such that the corresponding IET T satisfies the condition SUDC. Then, since the SUDC has full measure by Theorems 3.8 and 5.6, one can show by definition of the measure class on $\mathscr{U}_{min}$ (see for example [63]) that the set of locally Hamiltonian flows satisfying the condition SUDC has full measure in $\mathscr{U}_{min}$ (in the sense of § 2.1.2).

In view of Propositions 2.4 and 2.5, we equivalently need to prove the dichotomy between ergodicity and reducibility for the skew product map T_{φ_f}. Furthermore, we know from Proposition 4.1 that the cocycle φ_f is such that $\varphi_f \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, $\partial_\pi(\varphi_f) = 0$, $g'_{\varphi_f} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\mathcal{AS}(\varphi_f) = 0$.

Definition of the subspace K. Let us consider the linear operator $\mathfrak{H} : C^{2+\epsilon}(M) \to F \simeq \mathbb{R}^g$ given by $\mathfrak{H}(f) = \mathfrak{h}(\varphi_f)$. As the composition of two bounded operators, it is also bounded. Let $K := \ker \mathfrak{H} \subset C^{2+\epsilon}(M)$. Then K is a closed subspace of codimension g (the genus of M).

Ergodicity. Suppose that $f \in K$ and $\sum_{\sigma\in\mathrm{Fix}(\psi_{\mathbb{R}})} |f(\sigma)| > 0$. By Proposition 4.1, $\varphi \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, $\mathcal{L}(\varphi_f) > 0$, $g'_{\varphi_f} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\mathcal{AS}(\varphi_f) = 0$. As $f \in K$, we additionally have $\mathfrak{h}(\varphi_f) = 0$. In view of Theorem 8.1, this gives the ergodicity of the skew product T_{φ_f}. By Proposition 2.4, we have the ergodicity of the extended flow $\Phi^f_{\mathbb{R}}$.

Reducibility. Suppose that $f \in K$ and $\sum_{\sigma\in\mathrm{Fix}(\psi_{\mathbb{R}})} |f(\sigma)| = 0$. By Proposition 4.1, $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ with $\partial_\pi(\varphi_f) = 0$ and $\varphi'_f \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$. As $f \in K$, we additionally have $\mathfrak{h}(\varphi_f) = 0$. In view of Theorem 8.12, φ_f is a coboundary with a bounded transfer map having at most countably many discontinuities. By Proposition 2.5, this gives the reducibility of the extended flow $\Phi^f_{\mathbb{R}}$. □

8.2.4. *Equidistribution of the error in the symmetric case.* We can now conclude also the proof of Theorem 1.3, by proving that in this case err_b is uniformely bounded and deducing from ergodicity the equidistribution statement for the singular cocycles as well as the error term.

Proof of the second part of Main Theorem 1.3. Suppose that $\psi_{\mathbb{R}} \in \mathcal{U}_{min}$ is minimal and satisfies the SUDC. Let $f : M \to \mathbb{R}$ be any $C^{2+\epsilon}$-observable.

Boundedness of the error. Let $f_{eb} : M \to \mathbb{R}$ be the map defined in (7.30). By construction (see (7.31) and (7.32)), f_{eb} is a $C^{2+\epsilon}$-map such that $f_{eb}(\sigma) = 0$ for all $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}})$ and $\mathfrak{h}(\varphi_{f_{eb}}) = 0$. By Proposition 4.1, we know furthermore that $\varphi_{f_{eb}} \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, $\varphi'_{f_{eb}} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\partial_\pi(\varphi_{f_{eb}}) = 0$. In view of Proposition 8.12, $\varphi_{f_{eb}}$ is a coboundary with a bounded transfer map having at most countably many discontinuities. By Proposition 2.5, this gives the reducibility of the extended flow $\Phi^{f_{eb}}_{\mathbb{R}}$, so there exists a continuous map

$u : M \to \mathbb{R}$ such that $\int_0^t f_{eb}(\psi_s x)\, ds = u(x) - u(\psi_t x)$. It follows that for every regular $x \in M$ and $t > 0$ we have

$$|err_b(f,t,x)| = \Big| \int_0^t f_{eb}(\psi_s x)\, ds \Big| \leq 2\|u\|_{\sup},$$

which completes the proof.

Equidistribution of the singular cocycles and the error term. Assume now in addition that $f \in C^{2+\epsilon}(M)$ is not identically zero on $\mathrm{Fix}(\psi_{\mathbb{R}})$. We will prove at the same time $err(f,t,x) = \int_0^t f_e(\psi_\tau x) d\tau$ and $u_\sigma(t,x) = \int_0^t \xi_\sigma(\psi_\tau x) d\tau$ are equidistributed on $\mathbb{R}$, in the sense of (1.9).

Let ξ be respectively $\xi = f_e$ or $\xi = \xi_\sigma$. We want to show that the assumptions of Theorem 8.1 hold for φ_ξ so that we can deduce that the skew product T_{φ_ξ} on $I \times \mathbb{R}$ is ergodic. In both cases, by Proposition 4.1, $\varphi_\xi, g'_{\varphi_\xi} \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and (by property (ii), since $\varphi_{\mathbb{R}} \in \mathcal{U}_{min}$) $\mathcal{AS}(\varphi_\xi) = 0$, $\partial_\pi(\varphi_\xi) = 0$. We claim furthermore that we also have that $\sum_{\sigma\in\mathrm{Fix}(\psi_{\mathbb{R}})} |\xi(\sigma)| > 0$ and therefore, also by Proposition 4.1, $\mathcal{L}(\varphi_\xi) > 0$. To see this for $\xi = f_e$, recall that in view of (7.26), $\sum_{\sigma\in\mathrm{Fix}(\psi_{\mathbb{R}})} |f_e(\sigma)| = \sum_{\sigma\in\mathrm{Fix}(\psi_{\mathbb{R}})} |f(\sigma)| > 0$. Furthermore, in view of (7.27), $\mathfrak{h}(\varphi_{f_e}) = 0$. For $\xi = \xi_\sigma$, on the other hand, recall that, by the definition of ξ_σ and (7.18), for every $\sigma \in \mathrm{Fix}(\psi_{\mathbb{R}})$ we have $\xi_\sigma(\sigma) = 1$ and $\mathfrak{h}(\xi_\sigma) = 0$.

Thus, since T satisfies the SUDC, all assumptions of Theorem 8.1 hold and we conclude that the skew product T_{φ_ξ} on $I \times \mathbb{R}$ is ergodic. It follows that also the skew product flow $(\Phi_t^\xi)_{t\in\mathbb{R}}$ on $M \times \mathbb{R}$ given by

$$\Phi_t^\xi(x,r) = \Big(\psi_t x, r + \int_0^t \xi(\psi_\tau x) d\tau\Big)$$

is ergodic. We now apply the ratio ergodic theorem to the ergodic flow $(\Phi_t^{f_e})_{t\in\mathbb{R}}$ and to the characteristic functions of the sets $I \times J_1$ and $I \times J_2$. Then for a.e. $(x,r) \in I \times \mathbb{R}$ for any pair of finite intervals $J_1, J_2 \subset \mathbb{R}$ we have

$$\frac{Leb\{t \in [0,T] : \int_0^t \xi(\psi_\tau x) d\tau \in J_1\}}{Leb\{t \in [0,T] : \int_0^t \xi(\psi_\tau x) d\tau \in J_2\}} = \frac{\int_0^T \chi_{I\times(J_1+r)}(\Phi_t^\xi(x,r))\, dt}{\int_0^T \chi_{I\times(J_2+r)}(\Phi_t^\xi(x,r))\, dt} \to \frac{|J_1 + r|}{|J_2 + r|} = \frac{|J_1|}{|J_2|}.$$

As $err(f,t,x) = \int_0^t f_e(\psi_\tau x) d\tau$ and $u_\sigma(t,x) = \int_0^t \xi_\sigma(\psi_\tau x) d\tau$, this gives the equidistribution of cocycles $t \mapsto err(f,t,x)$ and $t \mapsto u_\sigma(t,x)$ for a.e. $x \in M$. □

Appendix A.

In this Appendix we present the proofs of two auxiliary results, the ergodicity criterium (Proposition 8.4) in § A.1 and the cohomological reduction to piecewise linear cocycles (Theorem 8.9) in § A.2.

A.1. **Ergodicity criterium.** In this Appendix we prove the ergodicity criterium stated as Proposition 8.4. The proof repeats arguments from the proof of Propositions 5.1, 5.2 in [20] and is included for convenience.

Proof of Proposition 8.4. For simplicity assume that $|I| = 1$. First we show that there exists $C > 0$ such that

$$|\varphi^{(q_k)}(x) - \varphi^{(q_k)}(T^m x)| \leq C \text{ for all } 0 \leq m < p_k,\ x \in J^{(k)}. \tag{A.1}$$

Note that

$$|\varphi^{(q_k)}(x) - \varphi^{(q_k)}(T^m x)| = |\varphi^{(m)}(x) - \varphi^{(m)}(T^{q_k} x)| \leq \Big| \int_x^{T^{q_k} x} |(\varphi')^{(m)}(y)|\, dy \Big|.$$

Assume that $g_\varphi = 0$. In view of (SUDC2) in Proposition 5.6, for every $y \in I^{(k)}$ we have

$$|(\varphi')^{(m)}(y)| \leq \sum_{\alpha\in\mathcal{A}} \Big(\frac{|C_\alpha^+|}{\min_{0\leq i<m} |T^i y - l_\alpha|} + \frac{|C_\alpha^-|}{\min_{0\leq i<m} |T^i y - r_\alpha|} \Big) + M\mathcal{L}(\varphi)\|Q(k)\|.$$

As $x \in J^{(k)}$, by assumption (ii), there exists $c > 0$ such that

$$|T^i x - l_\alpha| \geq c/q_k,\ |T^i x - r_\alpha| \geq c/q_k,\ |T^i(T^{q_k} x) - l_\alpha| \geq c/q_k,\ |T^i(T^{q_k} x) - r_\alpha| \geq c/q_k$$

for all $\alpha \in \mathcal{A}$ and $0 \leq i < p_k$. As $x, T^{q_k} x \in I^{(k)}$, it follows that

$$|T^i y - l_\alpha| \geq c/q_k,\ |T^i y - r_\alpha| \geq c/q_k \text{ for all } y \in [x, T^{q_k} x].$$

In view of (3.14), this gives

$$\begin{aligned} \Big| \int_x^{T^{q_k} x} |(\varphi')^{(m)}(y)|\, dy \Big| &\leq |x - T^{q_k} x| \mathcal{L}(\varphi)(q_k/c + M\|Q(k)\|) \\ &\leq |I^{(k)}| \|Q(k)\| (M + 1/c)\mathcal{L}(\varphi) \leq \kappa(M + 1/c)\mathcal{L}(\varphi). \end{aligned}$$

Suppose that $g_\varphi \neq 0$. As $x, T^{q_k}x \in I^{(k)}$, we have that $\{T^i[x, T^{q_k}x] : 0 \leq i < m\}$ is a tower of intervals. Hence

$$|g_\varphi^{(m)}(x) - g_\varphi^{(m)}(T^{q_k}x)| \leq \sum_{0\leq i<m} |g_\varphi(T^i x)| - g_\varphi(T^i(T^{q_k}x))| \leq \operatorname{Var} g_\varphi.$$

This gives (A.1). Therefore, for every $0 \leq i < p_k$ we have

$$\int_{T^iJ^{(k)}} |\varphi^{(q_k)}(x)|\,dx \leq \int_{J^{(k)}} |\varphi^{(q_k)}(x)|\,dx + |J^{(k)}|C = \int_{J^{(k)}} |S(k)\varphi(x)|\,dx + |J^{(k)}|C.$$

Hence

$$\int_{\Xi_k} |\varphi^{(q_k)}(x)|\,dx \leq p_k \int_{I^{(k)}} |\varphi^{(q_k)}(x)|\,dx + p_k|J^{(k)}|C = \frac{1}{|I^{(k)}|}\int_{I^{(k)}} |S(k)\varphi(x)|\,dx + C.$$

In view of assumption (i), this gives the left condition in (8.1).

For every $0 \leq l < p_k$ let $[a_l, b_l] = J_l^{(k)}$. Repeating some integration by parts arguments from the proof of Proposition 5.2 in [20], we have

$$\Big|\int_{J_l^{(k)}} e^{2\pi s\varphi^{(q_k)}(x)}\,dx\Big| \leq \frac{1}{|s|}\Big(\frac{2}{\min_{x\in[a_l,b_l]} |(\varphi')^{(q_k)}(x)|} + \operatorname{Var}|_{[a_l,b_l]} \frac{1}{(\varphi')^{(q_k)}}\Big).$$

In view of (iii), it follows that

$$\Big|\int_{J_l^{(k)}} e^{2\pi s\varphi^{(q_k)}(x)}\,dx\Big| \leq \frac{1}{|s|}\Big(\frac{2}{cq_k} + \frac{1}{c^2q_k^2}\operatorname{Var}|_{[a_l,b_l]}(\varphi')^{(q_k)}\Big) \leq \frac{1}{|s|}\Big(\frac{2}{cq_k} + \frac{1}{c^2q_k^2}\sum_{0\leq i<q_k}\operatorname{Var}|_{T^i[a_l,b_l]}\varphi'\Big).$$

By (ii), $\{T^i[a_l, b_l] : 0 \leq i < q_k\}$ is a tower of intervals and each level interval $T^i[a_l, b_l]$ is distant from the set $End(T)$ by at least c/q_k. Recall that

$$\varphi'(x) = -\sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^+}{\{x - l_\alpha\}} + \sum_{\alpha\in\mathcal{A}} \frac{C_\alpha^-}{\{r_\alpha - x\}} + g_\varphi'(x).$$

Moreover,

$$\sum_{0\leq i<q_k} \operatorname{Var}|_{T^i[a_l,b_l]} \frac{1}{\{x - l_\alpha\}} = \operatorname{Var}_{[c/q_k,1]} \frac{1}{x} \leq \frac{q_k}{c},$$

$$\sum_{0\leq i<q_k} \operatorname{Var}|_{T^i[a_l,b_l]} \frac{1}{\{r_\alpha - x\}} = \operatorname{Var}_{[0,1-c/q_k,1]} \frac{1}{1-x} \leq \frac{q_k}{c}$$

and

$$\sum_{0\leq i<q_k} \operatorname{Var}|_{T^i[a_l,b_l]} g_\varphi' \leq \operatorname{Var} g_\varphi'.$$

It follows that for every $0 \leq l < p_k$,

$$\Big|\int_{J_l^{(k)}} e^{2\pi s\varphi^{(q_k)}(x)}\,dx\Big| \leq \frac{1}{|s|}\Big(\frac{2}{cq_k} + \frac{1}{c^2q_k^2}\Big(\mathcal{L}(\varphi)\frac{q_k}{c} + \operatorname{Var}(g_\varphi')\Big)\Big).$$

As

$$Leb\Big(\Xi_k \setminus \bigcup_{0\leq l<p_k} J_l^{(k)}\Big) = \sum_{0\leq l<p_k} Leb(T^lJ^{(k)} \setminus J_l^{(k)}) \leq \frac{2}{3}\sum_{0\leq l<p_k} |T^lJ^{(k)}| = \frac{2}{3}Leb(\Xi_k),$$

this yields

$$\Big|\int_{\Xi_k} e^{2\pi s\varphi^{(q_k)}(x)}\,dx\Big| \leq \frac{2}{3}Leb(\Xi_k) + \frac{1}{|s|}\Big(\frac{2}{c} + \frac{1}{c^2}\Big(\frac{\mathcal{L}(\varphi)}{c} + \operatorname{Var}(g_\varphi')\Big)\Big),$$

which gives the right condition in (8.1). By Proposition 8.2, we have the ergodicity of T_φ. □

A.2. **Cohomological reduction.** In this Appendix we present the proof of the cohomological reduction stated as Theorem 8.9. We will assume throughout that T satisfies the UDC. For simplicity will also assume that $|I| = 1$. Let us denote by

$$\mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha) := \{\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha), \text{ such that } \varphi' \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha) \text{ and } \mathfrak{h}(\varphi') = 0\}.$$

Outline of the proof. We will show first of all that every $\varphi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ with $\varphi' \in \mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ can be modified by a piecewise linear map such that its modification is in $\mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$, by showing that one can subtract a map whose derivative is $\mathfrak{h}(\varphi')$ (see Steps 1 and 2 of the proof of Theorem 8.9 below). The next step of the proof is to apply the correction by a piecewise constant function described in § 6 (see Step 3 of the proof of Theorem 8.9). We then show that, after this further correction, the resulting map $\widetilde{\varphi}$ is a coboundary. We will show more precisely that $\|S(k)\widetilde{\varphi}\|_{\sup}$ decays exponentially (see Theorem A.1). Then standard arguments based on decompositions of Birkhoff sums (see § 7.1.2) and the Gottschalk-Hedlund theorem yield that $\widetilde{\varphi}$ is a coboundary (see Step 4 of the proof of Theorem 8.9).

The proof of Theorem A.1 (namely of exponential decay of $\|S(k)\widetilde{\varphi}\|_{\sup}$) is similar to the proof of Theorem 6.1 in § 6, or more precisely to the the proof of sub-exponential growth of $\|S(k)\widetilde{\varphi}\|_{L^1(I^{(k)})}/|I^{(k)}|$ (see in particular (6.4) in the statement of Theorem 6.1). One of the key arguments in this proof was showing that $\mathcal{LV}(S(k)\varphi)$ was bounded (or had sub-exponential growth in the non-symmetric case). Here, we will have a stronger input, namely the exponential decay of $\mathcal{LV}(S(k)\varphi)$: indeed, for every $\varphi \in \mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, since φ is piecewise absolutely continuous, we have that $\mathcal{LV}(S(k)\varphi) = \mathrm{Var}(S(k)\varphi)$ and therefore, in view of Theorem 6.1 (applied to φ') and the control of the L^1 norm via $\|\cdot\|_{\mathcal{LV}}$ given by (4.3), for every $k \geq 1$,

$$\mathcal{LV}(S(k)\varphi) = \mathrm{Var}(S(k)\varphi) = \|S(k)(\varphi')\|_{L^1(I^{(k)})} \leq C|I^{(k)}|C'_k(T)\|\varphi'\|_{\mathcal{LV}}. \tag{A.2}$$

Since $|I^{(k)}|$ decays exponentially, this shows that $\mathcal{LV}(S(k)\varphi)$ decays exponentially. Exploiting this exponential decay, analyzing its effect on all inequalities used in § 6, we will prove the exponential decay of $\|S(k)\widetilde{\varphi}\|_{\sup}$. Differently than in § 6, though, instead of the L^1-norm, we have now to always use the sup-norm. This requires a detailed and patient analysis of all steps used in § 6 in this new context, which is performed for example in the proofs of Lemmas A.2 and A.3 below.

We begin by stating and proving the following exponential decay result.

Theorem A.1. *Assume that T satisfies the UDC. Suppose that $\varphi \in \mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$, $\partial_\pi(\varphi) = 0$ and $\mathfrak{h}(\varphi) = 0$. Then*

$$\|S(k)\varphi\|_{\sup} = O(e^{-\lambda k}).$$

The proof of Theorem A.1 will follow from combining the following three Lemmas (Lemma A.2, Lemma A.3 and Lemma A.4). The first is an improved estimate of the growth of the image $P^{(k)}\varphi$ of the correcting operators $P^{(k)}$ (introduced in § 6) when $\varphi \in \mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$ and $\partial_{\pi^{(0)}}(\varphi) = 0$.

Lemma A.2. *The correcting operator $P^{(k)} : \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha)/\Gamma^{(k)}_s$ is such that, for every $\varphi \in \mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$ with $\partial_{\pi^{(0)}}(\varphi) = 0$,*

$$\|P^{(k)}(S(k)\varphi)\|_{\sup/\Gamma^{(k)}_s} \leq C\|\varphi'\|_{\mathcal{LV}}\, W_k, \quad \text{where} \quad W_k := \sum_{r\geq k} \|Q_s(k, r+1)\|\|Z(r+1)\||I^{(r)}|C'_r(T). \tag{A.3}$$

Proof. Recall that $P^{(k)}$ is given by $P^{(k)} = U^{(k)} \circ P^{(k)}_0 - \Delta^{(k)}$. Let us first give a preliminary estimate for the modifying operator $\Delta^{(k)} : \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I^{(k)}_\alpha) \to H(\pi^{(k)})/\Gamma^{(k)}_s$ starting from the definition of $\Delta^{(k)}$ as the series given by (6.20). Let $\varphi \in \mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}} I^{(0)}_\alpha)$ with $\partial_{\pi^{(0)}}(\varphi) = 0$.

Step 1. Estimates of $\Delta^{(k)}\varphi$. To estimate the series (6.20) (with $S(k)(\varphi)$ instead than φ), for each fixed $r \geq k$ we need to estimate

$$(S_\flat(k, r+1))^{-1} \circ U^{(r+1)} \circ \mathcal{M}^{(r+1)}_H \circ S(r, r+1) \circ P^{(r)}_0 \circ S(r)(\varphi).$$

Let us start from right to left, by estimating first the action of $P^{(r)}_0$ on $S(r)(\varphi)$, then that one of $\mathcal{M}^{(r+1)}_H \circ S(r, r+1)$ and finally applying and estimating $(S_\flat(k, r+1))^{-1} \circ U^{(r+1)}$.

Step 1 (i). The action of $P^{(r)}_0$. Recall that (by Definition 8) $P^{(r)}_0 = I - p_{H(\pi^{(r)})} \circ \mathcal{M}^{(r)}$, where I is the identity operator and $\mathcal{M}^{(r)}$ the preliminary correction given by subtracting the mean in each $I^{(r)}_\alpha$. Since $S(r)(\varphi) \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I^{(r)}_\alpha)$ is piecewise absolutely continuous,

$$\|S(r)(\varphi) - \mathcal{M}^{(r)}(S(r)(\varphi))\|_{\sup} \leq \mathrm{Var}(S(r)(\varphi)).$$

Using first this estimate together with the control of the projection by the boundary operator $\|h - p_{H(\pi^{(r)})}h\| \leq C_{\mathcal{G}}\|\partial^{(r)}_\pi h\|$ given by Lemma 3.4 (see in particular (3.13)), then the comparision between $\partial_{\pi^{(r)}}$ and $\partial_{\pi^{(r)}} \circ \mathcal{M}^{(r)}$ given by (4.23) and finally the estimate (A.2) of the variance together with the invariance of the boundary (5.7) and the assumption that $\partial_{\pi^{(r)}}(S(r)\varphi) = \partial_{\pi^{(0)}}(\varphi) = 0$, we get the following chain of inequalities:

(A.4)
$$\begin{aligned}
\|P^{(r)}_0 \circ S(r)(\varphi)\|_{\sup} &\leq \|S(r)(\varphi) - \mathcal{M}^{(r)}(S(r)(\varphi))\|_{\sup} + \|\mathcal{M}^{(r)}(S(r)(\varphi)) - p_{H(\pi^{(r)})}\mathcal{M}^{(r)}(S(r)(\varphi))\|_{\sup} \\
&\leq \mathrm{Var}(S(r)(\varphi)) + C_{\mathcal{G}}\|\partial_{\pi^{(r)}}\mathcal{M}^{(r)}(S(r)(\varphi))\| \\
&\leq (1 + 2dC_{\mathcal{G}})\,\mathrm{Var}(S(r)(\varphi)) + C_{\mathcal{G}}\|\partial_{\pi^{(r)}}(S(r)\varphi)\| \\
&\leq C'|I^{(r)}|C'_r(T)\|\varphi'\|_{\mathcal{LV}}.
\end{aligned}$$

Step 1(ii). The action of $\mathcal{M}^{(r+1)}_H \circ S(r, r+1)$. In view of the initial correction estimates of Lemma 6.4 (in particular (6.8)), the L^1-norm of special Birkhoff sums estimate (5.2) and the interval lenght control in terms of cocycle matrix norms given by (3.4), for every $\phi \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(r)})$,

$$\begin{aligned}\|\mathcal{M}_H^{(r+1)} \circ S(r, r+1)\phi\| &\leq \frac{\kappa\sqrt{d}}{|I^{(r+1)}|}\|S(r, r+1)\phi\|_{L^1(I^{(r+1)})}\\ &\leq \frac{\kappa\sqrt{d}}{|I^{(r+1)}|}\|\phi\|_{L^1(I^{(r)})} \leq \kappa\sqrt{d}\frac{|I^{(r)}|}{|I^{(r+1)}|}\|\phi\|_{\sup} \leq \kappa\sqrt{d}\|Z(r+1)\|\|\phi\|_{\sup}.\end{aligned}$$

Step 1(iii). The action of $(S_\flat(k, r+1))^{-1} \circ U^{(r+1)}$. Since $\|U^{(r+1)}\| = 1$, by (A.4), this gives (applied to $\phi = P_0^{(r)} \circ S(r)(\varphi)$)

$$\begin{aligned}&\|(S_\flat(k, r+1))^{-1} \circ U^{(r+1)} \circ \mathcal{M}_H^{(r+1)} \circ S(r, r+1) \circ P_0^{(r)} \circ S(k, r)(S(k)\varphi)\|\\ &\leq \kappa\sqrt{d}C'\|Q_s(k, r+1)\|\|Z(r+1)\||I^{(r)}|C_r'(T)\|\varphi'\|_{\mathcal{LV}}.\end{aligned}$$

As $\Delta^{(k)}(S(k)\varphi)$ is the sum of the series (6.20), it follows that

$$\|\Delta^{(k)}(S(k)\varphi)\|_{\sup/\Gamma_s^{(k)}} \leq \kappa\sqrt{d}C'W_k\|\varphi'\|_{\mathcal{LV}}. \tag{A.5}$$

Step 2. Estimates of $P^{(k)}\varphi$. We can now estimate $P^{(k)} : \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)}) \to \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})/\Gamma_s^{(k)}$ recalling that $P^{(k)} = U^{(k)} \circ P_0^{(k)} - \Delta^{(k)}$. As $\|U^{(k)}\| = 1$, in view of (A.4), if $\varphi \in \mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha)$ and $\partial_\pi(\varphi) = 0$ then

$$\|U^{(k)} \circ P_0^{(k)}(S(k)\varphi)\|_{\sup/\Gamma_s^{(k)}} \leq \|P_0^{(k)} \circ S(k)(\varphi)\|_{\sup} \leq C'|I^{(k)}|C_k'(T)\|\varphi'\|_{\mathcal{LV}} \leq C'W_k\|\varphi'\|_{\mathcal{LV}}.$$

Together with (A.5), this gives the desired estimate and proves the Lemma. □

Lemma A.3. *Under the assumptions of Theorem A.1, for any $k \geq 0$,*

$$\|S(k)\varphi\|_{\sup} \leq C\big(\|\varphi'\|_{\mathcal{LV}}\, V_k + \|Q_s(k)\|\|\varphi\|_{\sup}\big),$$

where V_k is given by the following series

$$V_k = \sum_{0\leq l\leq k}\|Q_s(l, k)\|\big(W_l + \|Z(l)\|W_{l-1}\big),$$

in which $W_{-1} := 0$ by convention and the series W_l for $l \geq 0$ is defined in (A.3) *of Lemma A.2.*

Proof. By the definition of the operator $\mathfrak{h}$ (see (6.29)), since $\mathfrak{h}(\varphi) = 0$, we have that $U^{(0)}(\varphi) = P^{(0)}(\varphi)$. In view of the equivariance described by Lemma 6.7, it follows that

$$U^{(k)} \circ S(k)\varphi = S_\flat(k) \circ U^{(0)}\varphi = S_\flat(k) \circ P^{(0)}\varphi = P^{(k)} \circ S(k)\varphi.$$

Therefore, by Lemma A.2, we have

$$\|U^{(k)} \circ S(k)\varphi\|_{\sup/\Gamma_s^{(k)}} = \|P^{(k)}(S(k)\varphi)\|_{\sup/\Gamma_s^{(k)}} \leq CW_k\|\varphi'\|_{\mathcal{LV}}.$$

It follows that for every $k \geq 0$ there exists $\varphi_k \in \mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(k)})$ and $s_k \in \Gamma_s^{(k)}$ such that

$$S(k)\varphi = \varphi_k + s_k \text{ and } \|\varphi_k\|_{\sup} \leq CW_k\|\varphi'\|_{\mathcal{LV}}. \tag{A.6}$$

Setting $s_0 := \Delta s_0$ and $\Delta s_{k+1} = s_{k+1} - Z(k+1)s_k$ for any $k \geq 1$, since for $s_k \in \Gamma^{(k)}$ we have that $S(k, k+1)s_k = Z(k+1)s_k$, we get

$$\Delta s_{k+1} = s_{k+1} - S(k, k+1)s_k = (S(k+1)\varphi - \varphi_{k+1}) - S(k, k+1)(S(k)\varphi - \varphi_k) = -\varphi_{k+1} + S(k, k+1)\varphi_k.$$

Therefore, by (A.6),

$$\begin{aligned}\|\Delta s_{k+1}\|_{\sup} &= \|\varphi_{k+1} - S(k, k+1)\varphi_k\|_{\sup} \leq \|\varphi_{k+1}\|_{\sup} + \|S(k, k+1)\varphi_k\|_{\sup}\\ &\leq \|\varphi_{k+1}\|_{\sup} + \|Z(k+1)\|\|\varphi_k\|_{\sup} \leq C(W_{k+1} + \|Z(k+1)\|W_k)\|\varphi'\|_{\mathcal{LV}}\end{aligned}$$

and, since by definition $\Delta s_0 = s_0 = \varphi - \varphi_0$,

$$\|\Delta s_0\|_{\sup} = \|\varphi - \varphi_0\|_{\sup} \leq \|\varphi\|_{\sup} + CW_0\|\varphi'\|_{\mathcal{LV}}.$$

Since $s_k = \sum_{0\leq l\leq k} Q(l, k)\Delta s_l$ and $\Delta s_l \in \Gamma_s^{(l)}$, setting $W_{-1} = 0$, we have

$$\begin{aligned}\|s_k\|_{\sup} &\leq \sum_{0\leq l\leq k}\|Q(l, k)\Delta s_l\|_{\sup} \leq \sum_{0\leq l\leq k}\|Q_s(l, k)\|\|\Delta s_l\|_{\sup}\\ &\leq \|Q_s(k)\|\|\varphi\|_{\sup} + C\sum_{0\leq l\leq k}\|Q_s(l, k)\|(W_l + \|Z(l)\|W_{l-1})\|\varphi'\|_{\mathcal{LV}}.\end{aligned}$$

In view of (A.6), it follows that

$$\|S(k)\varphi\|_{\sup} \leq \|\varphi_k\|_{\sup} + \|s_k\| \leq \|Q_s(k)\|\|\varphi\|_{\sup} + 2C\sum_{0\leq l\leq k}\|Q_s(l, k)\|(W_l + \|Z(l)\|W_{l-1})\|\varphi'\|_{\mathcal{LV}},$$

which completes the proof. □

Lemma A.4. *Suppose that T satisfies the UDC. Then for every $0<\tau<(\lambda_1-\lambda)/5$ we have $V_k=O(e^{-\lambda k})$.*

Proof. In view of (3.27) in Proposition 3.9 and $|I^{(r)}|\leq\kappa\|Q(r)\|^{-1}=O(e^{-\lambda_1 r})$ (see (3.14) and (UDC3)), we have

$$\begin{aligned}W_k&=O\Big(\sum_{r\geq k}\|Q_s(k,r+1)\|\|Z(r+1)\||I^{(r)}|C'_r(T)\Big)\\&=O\Big(\sum_{r\geq k}e^{-\lambda(r+1-k)}e^{4\tau r}e^{-\lambda_1 r}\Big)=O\Big(e^{-(\lambda_1-4\tau)k}\sum_{r\geq k}e^{-(\lambda+\lambda_1-4\tau)(r-k)}\Big)=O\big(e^{-(\lambda_1-4\tau)k}\big).\end{aligned}$$

By the definition of V_k, it follows that,

$$\begin{aligned}V_k&=O\Big(\sum_{0\leq l\leq k}\|Q_s(l,k)\|\big(e^{-(\lambda_1-4\tau)l}+\|Z(l)\|e^{-(\lambda_1-4\tau)(l-1)}\big)\Big)\\&=O\Big(\sum_{0\leq l\leq k}e^{-\lambda(k-l)}e^{\tau l}e^{-(\lambda_1-4\tau)l}\Big)=O\Big(e^{-\lambda k}\sum_{0\leq l\leq k}e^{-(\lambda_1-\lambda-5\tau)l}\Big)=O\big(e^{-\lambda k}\big).\end{aligned}$$

□

Proof of Theorem A.1. The proof follows immediately by combining Lemma A.3 and Lemma A.4, which show that

$$\|S(k)\varphi\|_{\sup}\leq C'\big(\|\varphi'\|_{\mathcal{LV}}\,e^{-\lambda k}+\|Q_s(k)\|\|\varphi\|_{\sup}\big),$$

Since also $\|Q_s(k)\|=O(e^{-\lambda k})$ by the UDC (see (UDC1) of Definition 3), we get that $\|S(k)\varphi\|_{\sup}=O(e^{-\lambda k})$. □

We can now also prove the cohomological reduction.

Proof of Theorem 8.9. Let us assume that T satisfies the UDC and that $\varphi\in\mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ and $\varphi'\in\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$. Fix any $0<\tau<\min\{(\lambda_1-\lambda)/5,\lambda\}$.

Step 1. First correction for the derivative to be in the kernel of $\mathfrak{h}$. Let $h_1=\mathfrak{h}(\varphi')\in H(\pi)$ and take any piecewise linear $\underline{\varphi}\in\mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ such that $\underline{\varphi}'=h_1$. Then $\mathfrak{h}((\varphi-\underline{\varphi})')=0$.

Step 2. Correction to be in $\mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$. Since $\mathfrak{h}((\varphi-\underline{\varphi})')=0$ by Step 1, Corollary 7.10 shows that the sum of jumps $s(\varphi-\underline{\varphi})=\int_I(\varphi-\underline{\varphi})'(x)\,dx=0$. By (4.2), it follows that

$$\sum_{\mathcal{O}\in\Sigma(\pi)}(\partial_\pi(\varphi-\underline{\varphi}))_{\mathcal{O}}=0.$$

Since the image of ∂_π consists of all vectors $(x_{\mathcal{O}})_{\mathcal{O}}$ such that $\sum_{\mathcal{O}\in\Sigma(\pi)}x_{\mathcal{O}}=0$ (see (3.11)), there exists $h_2\in\Gamma$ such that $\partial_\pi(h_2)=\partial_\pi(\varphi-\underline{\varphi})$. We claim that $\varphi-\underline{\varphi}-h_2$ belongs to $\mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$. To see this, notice first that $\varphi-\underline{\varphi}-h_2\in\mathrm{AC}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ and that $(\varphi-\underline{\varphi}-h_2)'=\varphi'-h_1\in\mathrm{LG}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$. Furthermore

$$\partial_\pi(\varphi-\underline{\varphi}-h_2)=0,\quad\mathfrak{h}((\varphi-\underline{\varphi}-h_2)')=\mathfrak{h}((\varphi-\underline{\varphi})')=0,$$

so $\varphi-\underline{\varphi}-h_2\in\mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$.

Step 3. Last correction to be in the kernel of $\mathfrak{h}$. Let $h_3=\mathfrak{h}(\varphi-\underline{\varphi}-h_2)\in H(\pi)$ and set

$$\widetilde{\varphi}:=\varphi-\underline{\varphi}-h_2-h_3.$$

Then $\widetilde{\varphi}\in\mathrm{AC}^{\mathfrak{h}}(\sqcup_{\alpha\in\mathcal{A}}I_\alpha)$ with $\mathfrak{h}(\widetilde{\varphi})=0$ and $\partial_\pi(\widetilde{\varphi})=\partial_\pi(\varphi-\underline{\varphi}-h_2)-\partial_\pi(h_3)=0$.

Step 4. Proof that $\widetilde{\varphi}$ is a coboundary. Given any every bounded function $\varphi:I\to\mathbb{R}$ and $n>0$, by decomposing the Birkhoff sums $\varphi^{(n)}$ into special Birkhoff sums (see for example [44, § 2.2.3]), we can get the estimate

$$\|\varphi^{(n)}\|_{\sup}\leq 2\sum_{l\in\mathbb{N}}\|Z(l+1)\|\|S(l)\varphi\|_{\sup},\tag{A.7}$$

As $0<\tau<\lambda$, in view of the UDC (in particular the estimate of $\|Z(l)\|$) and Theorem A.1, which gives that $\|S(l)\widetilde{\varphi}\|_{\sup}=O(e^{-\lambda l})$, it follows that

$$\|\widetilde{\varphi}^{(n)}\|_{\sup}=O\big(\sum_{l\in\mathbb{N}}\|Z(l+1)\|e^{-\lambda l}\big)=O\big(\sum_{l\in\mathbb{N}}e^{-(\lambda-\tau)l}\big)=O(1).$$

Applying Gottschalk-Hedlund type arguments (see [44, §3.4]), we obtain that $\widetilde{\varphi}$ is a coboundary with a bounded transfer map.

Step 5. Conclusive arguments. Let us now define $\psi:=\varphi-\widetilde{\varphi}$. By Step 4, ψ and φ differ by a coboundary, so they are cohomologous. Furthermore, since by definition of ψ and of $\widetilde{\varphi}$ (see Step 3)

$$\psi=\varphi-\widetilde{\varphi}=\underline{\varphi}+h_2+h_3,$$

and $\underline{\varphi}$, h_2 and h_3 are all piecewise linear (actually piecewise constant in the case of h_2 and h_3) functions (by construction, see Step 1 and Step 2), we see that ψ is piecewise linear. Furthermore, since by construction $\mathfrak{h}(\widetilde{\varphi}) = 0$ and $\partial_\pi(\widetilde{\varphi})$ (in view of Step 3), we have that

$$\mathfrak{h}(\psi) = \mathfrak{h}(\varphi) - \mathfrak{h}(\widetilde{\varphi}) = \mathfrak{h}(\varphi) \quad \text{and} \quad \partial_\pi(\psi) = \partial_\pi(\varphi) - \partial_\pi(\widetilde{\varphi}) = \partial_\pi(\varphi).$$

Finally, Theorem A.1 shows that $\|S(k)(\varphi - \psi)\|_{\sup} = \|S(k)\widetilde{\varphi}\|_{\sup}$ decay exponentially. This completes the proof. □

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