DIRECTIONAL LOCALIZATION OF LIGHT RAYS IN A PERIODIC ARRAY OF RETRO-REFLECTOR LENSES

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Abstract. We show that vertical light rays in almost every periodic array of Eaton lenses do not leave certain strips of bounded width. The light rays are traced by leaves of a non-orientable foliation on a singular plane. We study the flow defined by the induced foliation on the orientation cover of the singular plane. The behavior of that flow and ultimately our claim for the light rays is based on an analysis of the Teichmüller flow and the Kontsevich-Zorich cocycle on the moduli space of two branched, two sheeted torus covers in genus two.

1. Introduction

Trajectories of light can be controlled by systems of mirrors (like in billiard models) and also by changing the refractive index (RI) of a lens. In this note we will deal with the so called Eaton lens. This is a retroreflector lens that reflects rays of light back to their sources. More precisely, the Eaton lens is a round lens (of radius, say $R > 0$) where the RI varies from 1 to infinity and it is given by $RI = \sqrt{\frac{2R}{r} - 1}$ in polar coordinates. We assume that the refraction index outside the lens is equal to 1. The RI is not defined at the center of the lens and goes to infinity when approaching this singular point. The direction of the light motion is reversed (cf. [8]) after passing through the lens, see Figure 1. Next let us consider

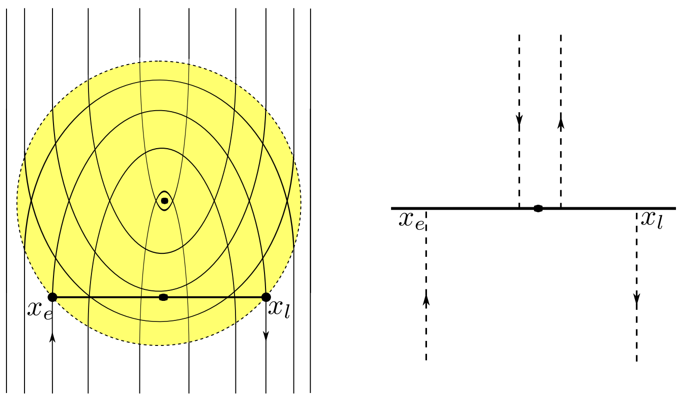

Figure 1. Eaton lens and its flat counterpart

a system of identical Eaton lenses (of radius $R > 0$) that are arranged on the plane $\mathbb{R}^2$ so that the centers are placed at the points of a lattice $\Lambda \subset \mathbb{R}^2$. We say that a lattice Λ is R-admissible if the circles of radius R centered at the lattice points of Λ are pairwise disjoint. We will denote such admissible system of Eaton lenses by $L(\Lambda, R)$, see Figure 2. Our purpose is to study the behavior of light orbits for

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such periodic system of lenses for different pairs of parameters Λ, R. First note that after a rotation we can assume that light rays running in the same direction are vertical and after rescaling we can assume that Λ is unimodular. Denote by $\mathscr{L}$ the space of unimodular lattices on $\mathbb{R}^2$ that can be identified with the moduli space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$. Let us consider the natural action of $SL(2,\mathbb{R})$ given by the left multiplication and denote by $\mu_{\mathscr{L}}$ the unique probability invariant measure on $\mathscr{L}$. For every $0 < R < 1/\sqrt{2\sqrt{3}}$ the set of unimodular R-admissible lattices is an open (non-empty, because the hexagonal unimodular lattice is R-admissible) subset of $\mathscr{L}$ and hence has positive measure. For $R \geq 1/\sqrt{2\sqrt{3}}$ the set of unimodular R-admissible lattices is empty, compare with the optimal circle packing problem. The main result of this note is the following.

Theorem 1.1. *For every $0 < R < 1/\sqrt{2\sqrt{3}}$ and for a.e. R-admissible lattice $\Lambda \in \mathscr{L}$ there exist constants $C = C(\Lambda, R) > 0$ and $\theta = \theta(\Lambda, R) \in S^1$, such that every vertical light ray in $L(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction θ.*

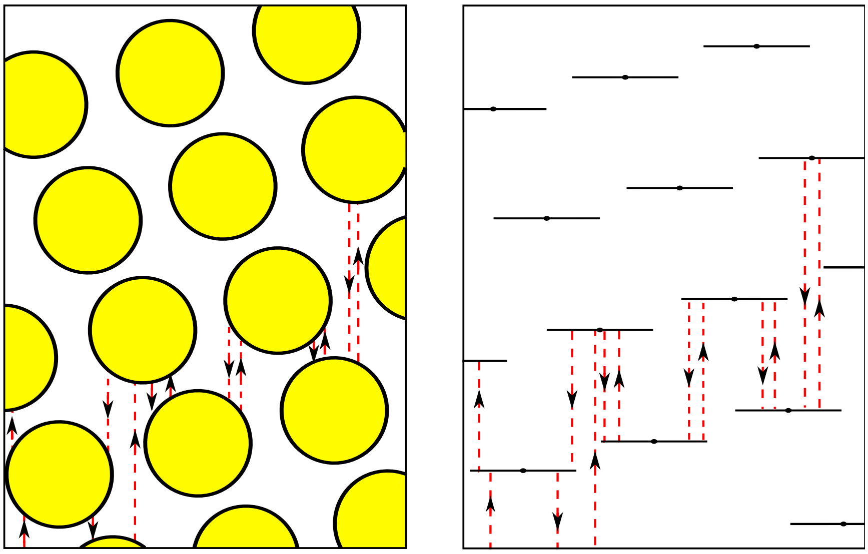


FIGURE 2. $L(\Lambda, R)$ and $F(\Lambda, R)$

Since we care only about the knowledge about orbits (not on the dynamics) of light rays, we can pass to a simpler model where round lenses are replaced by flat counterparts, i.e. vertical intervals of length $2R$ (called slits), see Figures 1 and 2. Such system of flat horizontal "lenses" of length $2R$ whose centers are placed at the points of a lattice Λ will be denoted by $F(\Lambda, R)$. The vertical light rays flow on $F(\Lambda, R)$ by vertical translation with unit speed until hitting the interior of a slit. Then a light ray in $F(\Lambda, R)$ is rotated by π around the center of the slit and runs vertically in the opposite direction until next impact, see the right part of Figure 1.

A vertical light ray in $L(\Lambda, R)$ entering to an Eaton lens at a point x_e leaves the lens at x_l so that the light is rotated by π around the center of the interval $[x_e, x_l]$ which is a horizontal chord of the lens, see the left part of Figure 1. There is one exception to this rule when a light ray approaches the center of the lens. Then the light ray does not leave the lens. We adopt the convention that such a light ray turns back at the center of the lens. Under this convention for every light orbit in $L(\Lambda, R)$ there is a corresponding orbit in $F(\Lambda, R)$ such that both orbits coincide outside the lattice of circles. Since inside the circles the distance between these orbits is bounded by $2R$, the distance between the whole corresponding light orbits in $L(\Lambda, R)$ and $F(\Lambda, R)$ is bounded by $2R$, as well.

In fact, we will deal with systems $F(\Lambda, R)$ for $R > 0$ and $\Lambda \in \mathscr{L}$ such that

$$\text{the slits in } F(\Lambda, R) \text{ are pairwise disjoint.} \tag{1}$$

Of course, R-admissibility of Λ implies this condition.

The problem of understanding the behavior of vertical light rays in $F(\Lambda, R)$ is reduced in Section 2 to the study of the vertical flow on a translation surface $\widetilde{M}(\Lambda, R)$ which is a $\mathbb{Z}^2$-cover of a compact translation surface $M(\Lambda, R)$ the union of two slit tori. The passage to the framework of translation surfaces allows us to exploit a powerful approach related to Teichmüller dynamics and Lyapunov exponents of the so called Kontsevich-Zorich cocycle. Exploiting the phenomenon of bounded deviation discovered by Zorich in [14, 15] we prove the following.

Theorem 1.2. *For every $R > 0$ and for a.e. lattice $\Lambda \in \mathscr{L}$ there exist constants $C = C(\Lambda, R) > 0$ and $\theta = \theta(\Lambda, R) \in S^1$, such that every vertical light ray in $F(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction θ.*

If Λ is R-admissible then every light orbit in $L(\Lambda, R)$ has a corresponding orbit in $F(\Lambda, R)$ so that the distance (in $\mathbb{R}^2$) between them is bounded by $2R$. Therefore, Theorem 1.1 follows directly from Theorem 1.2.

Outline of strategy. We have already seen that the Eaton lens dynamics converts to a dynamics on a plane with "slit reflectors" preserving the dynamical features which matter for our problem. The group $\mathbb{Z}^2$ acts on the slit plane by translations, so let us look at the simplest plane with a $\mathbb{Z}^2$-action, the complex plane. Take a curve $\widetilde{\gamma} : [0, 1] \to \mathbb{C}$ on the complex plane and its image $\gamma := p \circ \widetilde{\gamma} : [0, 1] \to \mathbb{T}^2$ on the the quotient $\mathbb{C} \overset{p}{\to} \mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ modulo $\mathbb{Z}^2$. To study properties of $\widetilde{\gamma}$, it is enough to look at the curve γ on $\mathbb{T}^2$ and study its lifts. Suppose we want to know the location of the endpoint $\widetilde{\gamma}(1)$ of a particular lift $\widetilde{\gamma}$. To do this we tile the complex plane by $\mathbb{Z}^2$ translates of the unit square $[0, 1)^2$, the fundamental domain representing $\mathbb{T}^2$ and fix an enumeration of the tiles by $\mathbb{Z}^2$. Given that $\widetilde{\gamma}(1) \in p^{-1}(\gamma(1))$, all we need to locate $\widetilde{\gamma}(1)$ is the integer coordinate of the tile containing it. That information can be derived from γ and the topology of $\mathbb{T}^2$. In fact, the coordinate of the tile containing $\gamma(1)$ is obtained from the tile coordinate of $\gamma(0)$ by adding the number of (oriented) crossings of γ with the images of horizontal tile edges and vertical tile edges on $\mathbb{T}^2$ with respect to p. This can be calculated as an algebraic intersection number of γ (after closing it up to a loop without generating new edge intersections) with the respective homology classes defined by vertical and horizontal edges.

Clearly the geometry and the geodesic dynamics on the slit plane, see Figure 2 on the right, differs from the euclidean one on $\mathbb{C}$ and is certainly more complex. Following the previous idea we want to calculate the intersection numbers of an orbit with the vectors generating the lattice translation symmetry of the slit plane. Those intersection numbers, or more precisely their asymptotic behavior, turn out to be sufficient to show our claims. Some technical difficulties emerge. One is that foliations on the quotient tori of the slit planes are not orientable, in the sense that they do not define flows. Generally for this kind of non orientable foliation on a surface, say S, there exists a unique double cover $M \to S$ the *orientation cover*, such that the pulled back foliation on M is orientable and hence defines a flow. While the choice of two homology classes on a torus for intersection calculations is more or less canonical, we need to isolate the right homology classes on the orientation cover.

Our general method to study the long term behavior of (vertical) leaves applies to various cases. To describe it we start with a less general case, in which the quotient torus, say T^2, carries an orientation preserving homeomorphism $\phi : T^2 \to T^2$ which is locally affine linear, a so called *affine homeomorphism*. Any affine homeomorphism, has constant derivative $D\phi \in \mathrm{PSL}_2(\mathbb{R})$ (except for a discrete set of points). We further need ϕ to be a *pseudo-Anosov* map, that is $D\phi$ is hyperbolic, which is the case if $|\mathrm{tr}(D\phi)| > 2$. A pseudo-Anosov has two eigendirections, defining the stable and unstable eigenfoliations on T^2. The leaves of the unstable foliation are expanded under the application of ϕ, while the leaves of the stable foliation are contracted. Up to a conjugation with a convenient affine linear transformation we may actually assume the vertical direction on T^2 is the stable eigendirection of a pseudo-Anosov on T^2.

Consider the homology class of a loop defined by closing up a segment of a vertical leaf. Then the key step in determining the intersection numbers is to calculate the induced map $\phi_* : H_1(T^2, \mathbb{R}) \to H_1(T^2, \mathbb{R})$ in homology. We can extract some of the information on the shape of light rays from ϕ_*. If ϕ_* is hyperbolic, the vertical direction is confined in a strip, moreover its stable eigendirection gives (together with some coordinate adjustment) the direction of the confining strip. This uses the phenomenon of bounded deviation discovered by Zorich, which applies to the vertical flow on the orientation cover $M_{T^2} \to T^2$. Note that the existence of pseudo-Anosov on T^2 implies the existence of pseudo-Anosovs on M_{T^2}.

Surfaces which have vertical foliations stabilized by a pseudo-Anosov are rather rare and in order to show the claim of Theorem 1.2 we need to consider a larger set of surfaces. The connection with the pseudo-Anosov case is made by the observation that orientation preserving homeomorphisms act on certain sets of flat surfaces with fixed topological data. Recall that we want to study dynamics on surfaces which are branched torus covers of degree two with two ramification points, see Figures 7 and 8, so let us consider the set of those covers. Up to isomorphism this set of torus covers defines a certain locus, say $\mathcal{M}$, in a stratum of moduli space. A point in $\mathcal{M}$ represents a surface equipped with a holomorphic one form determining the geometry and dynamics on the surface. We will move a surface around in $\mathcal{M}$, by applying the one parameter subgroup $(g_t)_{t\in\mathbb{R}}$ of $\mathrm{SL}_2(\mathbb{R})$, where $g_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$. The orbit of $M \in \mathcal{M}$ under $(g_t)_{t\in\mathbb{R}}$ is a *Teichmüller geodesic*. The locus $\mathcal{M}$ is connected, carries a natural orbifold structure and also admits a natural flow invariant ergodic finite measure equivalent to Lebesgue measure in local coordinates.

To study the asymptotic behavior of homology classes along Teichmüller geodesics one needs to replace the first homology group of a surface by a global object over $\mathcal{M}$. The *(homological) Hodge bundle* $\mathcal{H}$ over $\mathcal{M}$ is the bundle having as fiber over $M \in \mathcal{M}$ the first homology group of M. The object describing how homology classes change along geodesics and this is the Kontsevich-Zorich cocycle $G_t^{KZ} : \mathcal{H} \to \mathcal{H}$. Ergodicity of the invariant measure on $\mathcal{M}$ allows us to apply Oseledet's theorem and as a consequence the Kontsevich-Zorich cocycle has Lyapunov exponents. We only need particular Lyapunov exponents for a flow invariant sub-bundle characterized by the homology classes defining the infinite cover. Those Lyapunov exponents are known and were calculated by Bainbridge [1]. This strategy produces Theorem 1.2, i.e. the existence of a common trend for vertical light rays in $F(\Lambda, R)$ when the radius R is fixed and the choice of lattice Λ is random.

Our initial pseudo-Anosov example can be seen as a special case of this argument. In fact pseudo-Anosov maps appear as g_{t_0}, where $t_0 > 0$ is a period of a periodic Teichmüller geodesic. The previous strategy applies, if we restrict the respective objects defined over $\mathcal{M}$ to the closed geodesic and replace the Liouville measure on $\mathcal{M}$ by the flow invariant probability measure supported on the periodic orbit.

Applying this observation, we show in Section 6 that the vertical direction on $L(\Lambda, R)$, with $R = 1/3$ and $\Lambda = (1,0)\,\mathbb{Z} + ((3+\sqrt{21})/6, 1)\,\mathbb{Z}$, is a pseudo-Anosov eigendirection and so every vertical light ray in $L(\Lambda, R)$ is trapped in a band. We also show that every band has slope $-(\sqrt{21}+3\sqrt{5})/4$.

Remark 1. The authors believe that a stronger version of Theorem 1.1 is true, namely for every R-admissible lattice Λ and for almost every direction $\theta \in S^1$ all light rays on $L(\Lambda, R)$ in the direction θ are trapped in bands. However, we expect that its proof needs a much more advanced approach than used in the present work.

2. From lens lattices to translation surfaces

At the beginning of this section we briefly recall some basic notions related to translation surfaces and their $\mathbb{Z}^d$-covers. For further background material we refer the reader to [7, 9, 11, 12].

2.1. **Translation surfaces and their $\mathbb{Z}^d$-covers.** A translation surface is a pair (M, ω) where M is an orientable Riemann surface (not necessarily compact) and ω is a translation structure on M, that is a non-zero holomorphic 1-form also called *Abelian differential.* Let $\Sigma = \Sigma_\omega \subset M$ denote the set of zeros of ω which are also the *singular points* of the translation structure. For every $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ denote by $X_\theta = X_\theta^\omega$ the vector field in direction θ on $M \setminus \Sigma$, i.e. $\omega(X_\theta^\omega) = e^{i\theta}$. Then the corresponding directional flow $(\varphi_t^\theta)_{t\in\mathbb{R}} = (\varphi_t^{\omega,\theta})_{t\in\mathbb{R}}$, also known as *translation flow*, on $M \setminus \Sigma$ preserves the volume form $\nu_\omega = \frac{i}{2}\omega \wedge \overline{\omega} = \Re(\omega) \wedge \Im(\omega)$. We will use the notation $(\varphi_t^v)_{t\in\mathbb{R}}$ for the *vertical flow* (corresponding to $\theta = \frac{\pi}{2}$). If M is compact let us denote the area of (M, ω) by $A(\omega) = \nu_\omega(M)$.

Let (M, ω) be a compact connected translation surface. A $\mathbb{Z}^d$*-cover* of M is a surface $\widetilde{M}$ with a free totally discontinuous action of the group $\mathbb{Z}^d$ such that the quotient manifold $\widetilde{M}/\mathbb{Z}^d$ is homeomorphic to M. Then the projection $p : \widetilde{M} \to M$ is called a *covering map.* Denote by $\widetilde{\omega}$ the pullback of the form ω by the map p. Then $(\widetilde{M}, \widetilde{\omega})$ is a translation surface as well.

Remark 2. Up to isomorphism $\mathbb{Z}^d$-covers of M are in one-to-one correspondence with $H_1(M, \mathbb{Z})^d$. For any elements $\xi_1, \xi_2 \in H_1(M, \mathbb{Z})$ denote by $\langle \xi_1, \xi_2 \rangle$ the algebraic intersection number of ξ_1 with ξ_2.

The $\mathbb{Z}^d$-cover $\widetilde{M}_\gamma$ determined by $\gamma \in H_1(M, \mathbb{Z})^d$ has the following properties:

If σ is a closed curve in M, $[\sigma] \in H_1(M, \mathbb{Z})$ and

$$\bar{n} = (n_1, \ldots, n_d) := (\langle [\sigma], \gamma_1 \rangle, \ldots, \langle [\sigma], \gamma_d \rangle) \in \mathbb{Z}^d$$

then σ lifts to a path $\widetilde{\sigma} : [t_0, t_1] \to \widetilde{M}_\gamma$ such that $\widetilde{\sigma}(t_1) = \bar{n} \cdot \widetilde{\sigma}(t_0)$, where $\cdot$ denotes the action of $\mathbb{Z}^d$ on $\widetilde{M}_\gamma$.

2.2. **A translation surface associated to $F(\Lambda, R)$.** As in the case of rational billiards, let us consider a flow describing the dynamics of vertical light rays in $F(\Lambda, R)$. Let us label the slits of $F(\Lambda, R)$ by elements of $\mathbb{Z}^2$. Since the directions of such orbits are either positive or negative, the phase space of the flow consists of two copies of $F(\Lambda, R)$, one $F_+(\Lambda, R)$ for positive and one $F_-(\Lambda, R)$ for negative orbit segments. Denote by $\zeta_\pm : F_\pm(\Lambda, R) \to F(\Lambda, R)$ the map establishing a natural identification of each copy with $F(\Lambda, R)$. The light ray flow $(\widetilde{\varphi}_t)_{t\in\mathbb{R}}$ acts on each point of the phase space moving it vertically (in positive or negative direction) with unit speed until it hits the interior of a slit (flat lens), then the point is rotated around the center of the slit by the angle π and it changes from copy $F_\pm(\Lambda, R)$ to copy $F_\mp(\Lambda, R)$, see Figure 3. Let us rotate the copy $F_-(\Lambda, R)$ by the angle π around the center of $(0,0)$-th slit (denote this rotation by r_π), see Figure 4. Next glue the top (bottom) of the (m, n)-th slit in $F_+(\Lambda, R)$ to the bottom (top) of the (m, n)-th

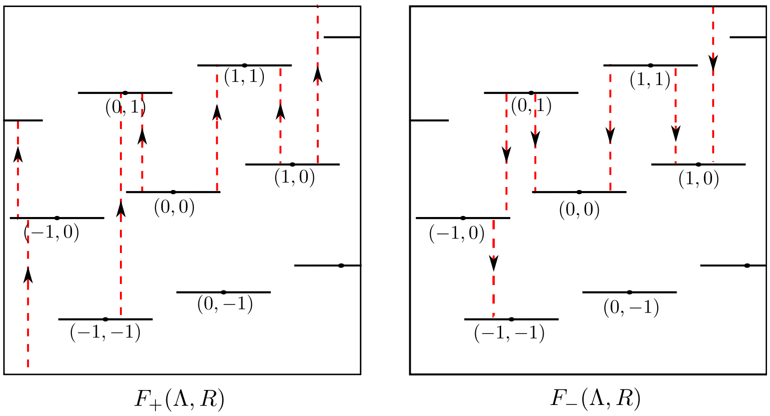


FIGURE 3. The phase space of the light rays flow

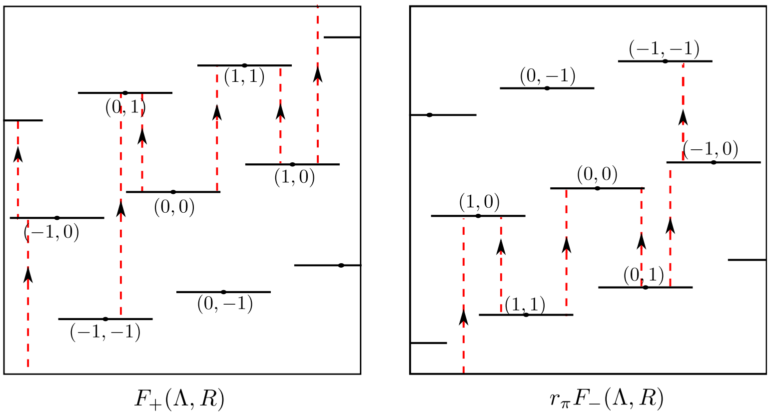


FIGURE 4. Components of the surface $\widetilde{M}(\Lambda,R)$

slit in $r_\pi F_-(\Lambda,R)$ for every $(m,n)\in\mathbb{Z}^2$. The resulting surface will be denoted by $\widetilde{M}(\Lambda,R)$. The surface $\widetilde{M}(\Lambda,R)$ carries a natural translation structure $\widetilde{\omega}$ whose restrictions to $F_+(\Lambda,R)$ and $r_\pi F_-(\Lambda,R)$ are defined by dz. Then the zeros of $\widetilde{\omega}$ (all of order one) arise from the ends of the slits. Moreover, the light rays flow $(\widetilde{\varphi}_t)_{t\in\mathbb{R}}$ regarded as a flow on $\widetilde{M}(\Lambda,R)$ is the translation flow in the vertical direction. Let us consider a free totally discontinuous action of Λ on $\widetilde{M}(\Lambda,R)$ given by

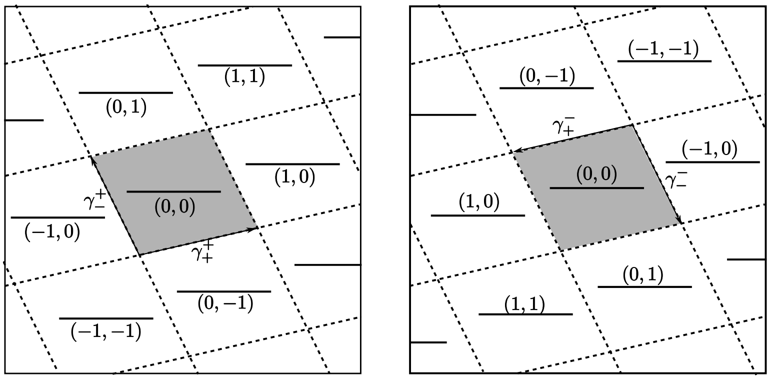
$$\lambda\cdot\widetilde{x}=\begin{cases}\zeta_+^{-1}(\zeta_+(\widetilde{x})+\lambda) & \text{if } \widetilde{x}\in F_+(\Lambda,R),\\ r_\pi\circ\zeta_-^{-1}((\zeta_-\circ r_\pi^{-1}(\widetilde{x}))+\lambda) & \text{if } \widetilde{x}\in r_\pi F_-(\Lambda,R).\end{cases}$$

Since this action preserves the form $\widetilde{\omega}$, we can consider the quotient translation surface which will be denoted by $M(\Lambda,R)$.

2.3. **A convenient representation of $M(\Lambda,R)$.** In this section we describe a representation of the translation surface $M(\Lambda,R)$ such that its $\mathbb{Z}^2$-cover $\widetilde{M}(\Lambda,R)$ has a convenient form.

Suppose that $R>0$ and $\Lambda\subset\mathbb{R}^2$ is a unimodular lattice satisfying (1). Then there exists a *positive* basis γ_+,γ_- of Λ, i.e. a basis with $\gamma_+\in\mathbb{R}_{++}$ and $\gamma_-\in\mathbb{R}_{-+}$, where

$$\mathbb{R}_{++}=\{(x,y)\in\mathbb{R}^2: x>0, y\geq 0\},\quad \mathbb{R}_{-+}=\{(x,y)\in\mathbb{R}^2: x\leq 0, y>0\}.$$



$F_+(\Lambda, R)$ $\qquad$ $r_\pi F_-(\Lambda, R)$

FIGURE 5. A domain for the Λ-action on $\widetilde{M}(\Lambda, R)$

Let us consider the centered parallelogram

$$P(\gamma_+, \gamma_-) = [-1/2, 1/2)\gamma_+ + [-1/2, 1/2)\gamma_-$$

generated by vectors $\gamma_+, \gamma_- \in \mathbb{R}^2$.

Lemma 2.1. *Let $R > 0$ and let Λ be a unimodular lattice so that* (1) *is valid. Then there exists a positive basis γ_+, γ_- of Λ such that*

(2) $\qquad [-R, R] \times \{0\}$ *is a subset of the interior of* $P(\gamma_+, \gamma_-)$.

Proof. Since the intersection of $P(\gamma_+, \gamma_-)$ and the line $\mathbb{R} \times \{0\}$ is a symmetric horizontal interval of length $1/\max(\gamma_2^+, \gamma_2^-)$ ($\gamma_2^\pm$ is the second coordinate of $\gamma_\pm$), we need find a positive basis γ_+, γ_- such that $0 \le \gamma_2^+, \gamma_2^- < \frac{1}{2R}$. Such a basis can be found using an Euclidean type algorithm starting from any positive basis a^0, b^0 of Λ. Indeed, let us consider the sequence $(a^n, b^n)_{n\ge 0}$ of positive bases of Λ defined inductively by:

$$a^{n+1} = a^n - b^n,\ b^{n+1} = b^n \text{ if } a^n - b^n \in \mathbb{R}_{++}$$
$$a^{n+1} = a^n,\ b^{n+1} = b^n - a^n \text{ if } b^n - a^n \in \mathbb{R}_{-+}.$$

If there exists $n \ge 0$ such that $a_2^n = 0$ then, by (1), $a_1^n > 2R$. Since Λ is unimodular, we have $1 = a_1^n b_2^n - a_2^n b_1^n = a_1^n b_2^n$. Therefore, $b_2^n < \frac{1}{2R}$ and hence a^n, b^n is a required positive basis.

Now suppose that $a_2^n > 0$ for every $n \ge 0$. By definition, the sequences $(a_2^n)_{n\ge 0}$ and $(b_2^n)_{n\ge 0}$ are non-increasing and hence $a_2^n \to a \ge 0$ and $b_2^n \to b \ge 0$. Since $(a_2^n)_{n\ge 0}$ and $(b_2^n)_{n\ge 0}$ are both positive, we have $a_2^{n+1} = a_2^n - b_2^n$ for infinitely many $n \ge 0$ and $b_2^{n+1} = b_2^n - a_2^n$ for infinitely many $n \ge 0$. It follows that $a = a - b$ and $b = b - a$, so $a = b = 0$. Therefore, we can find $n \ge 0$ with $a_2^n, b_2^n < \frac{1}{2R}$. Then a^n, b^n is a required positive basis. □

Suppose that γ_+, γ_- is a positive basis of Λ satisfying (2) and consider the action of the lattice Λ on $F(\Lambda, R)$ by translations. Then $P(\gamma_+, \gamma_-)$ is a fundamental domain for the Λ-action on $F(\Lambda, R)$ and $P(\gamma_+, \gamma_-)$ contains exactly one slit.

Let $\zeta : \widetilde{M}(\Lambda, R) \to F(\Lambda, R)$ be the map given by

$$\zeta(\widetilde{x}) = \begin{cases} \zeta_+(\widetilde{x}) & \text{if } \widetilde{x} \in F_+(\Lambda, R), \\ \zeta_- \circ r_\pi^{-1}(\widetilde{x}) & \text{if } \widetilde{x} \in r_\pi F_-(\Lambda, R). \end{cases} \tag{3}$$

Then ζ is two-to-one and, by the definition of the Λ-action on $\widetilde{M}(\Lambda, R)$, we have

$$\zeta(\lambda \cdot \widetilde{x}) = \zeta(\widetilde{x}) + \lambda \quad \text{for all} \quad \lambda \in \Lambda \text{ and } \widetilde{x} \in \widetilde{M}(\Lambda, R). \tag{4}$$

Therefore, the set

$$D := \zeta^{-1} P(\gamma_+, \gamma_-) \subset \widetilde{M}(\Lambda, R) \tag{5}$$

(see the shaded area in Figure 5) is a fundamental domain for the Λ-action on $\widetilde{M}(\Lambda, R)$. It follows that the compact translation surface $M(\Lambda, R)$ can be represented as the union of two identical tori glued along a horizontal slit of length $2R$ as in Figure 6.

Let $p : \widetilde{M}(\Lambda, R) \to M(\Lambda, R)$ denote the covering map and consider vectors γ_+^+, γ_+^-, γ_-^+, γ_-^- in $\widetilde{M}(\Lambda, R)$ as in Figure 5. Then $\zeta(\gamma_+^+) = \zeta(\gamma_+^-) = \gamma_+$ and $\zeta(\gamma_-^+) = \zeta(\gamma_-^-) = \gamma_-$. We will denote also by γ_+^+, γ_+^-, γ_-^+, γ_-^- the corresponding oriented curves in $\widetilde{M}(\Lambda, R)$. The projections $p(\gamma_+^+)$, $p(\gamma_+^-)$, $p(\gamma_-^+)$, $p(\gamma_-^-)$ are oriented loops in $M(\Lambda, R)$ whose homology classes generate the group $H_1(M(\Lambda, R), \mathbb{Z})$. Define

$$\gamma_1 := [p(\gamma_+^+)] + [p(\gamma_+^-)], \ \ \gamma_2 := [p(\gamma_-^+)] + [p(\gamma_-^-)] \in H_1(M(\Lambda, R), \mathbb{Z}),$$

see Figure 6. Now use the group isomorphism $\mathbb{Z}^2 \ni (m, n) \mapsto m\gamma_+ + n\gamma_- \in \Lambda$ to convert the Λ-action on $F(\Lambda, R)$ and $\widetilde{M}(\Lambda, R)$ into a $\mathbb{Z}^2$-action. By (4) we have

$$\zeta((m, n) \cdot \widetilde{x}) = (m, n) \cdot \zeta(\widetilde{x}) \quad \text{for all } \ (m, n) \in \mathbb{Z}^2 \text{ and } \ \widetilde{x} \in \widetilde{M}(\Lambda, R). \tag{6}$$

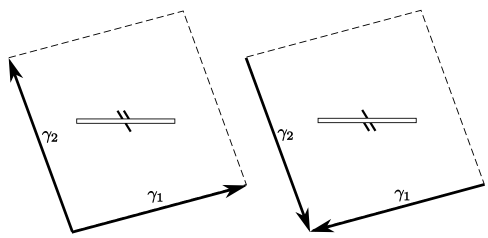


FIGURE 6. The surface $M(\Lambda, R)$

Lemma 2.2. *The translation surface $\widetilde{M}(\Lambda, R)$ is the $\mathbb{Z}^2$-cover of the compact translation surface $M(\Lambda, R)$ defined by $\gamma = (\gamma_2, -\gamma_1) \in H_1(M, \mathbb{Z})^2$.*

Proof. In view of Remark 2, we need to choose a finite set of oriented loops in $M(\Lambda, R)$ whose homology classes generate $H_1(M(\Lambda, R), \mathbb{Z})$ and show, that for every such loop $\sigma : [0, 1] \to M(\Lambda, R)$ any its lift $\widetilde{\sigma} : [0, 1] \to \widetilde{M}(\Lambda, R)$ fulfills

$$\widetilde{\sigma}(1) = \big(\langle [\sigma], \gamma_2 \rangle, -\langle [\sigma], \gamma_1 \rangle\big) \cdot \widetilde{\sigma}(0).$$

Of course, we will deal with the loops $p(\gamma_+^+)$, $p(\gamma_+^-)$, $p(\gamma_-^+)$, $p(\gamma_-^-)$ whose lifts γ_+^+, γ_+^-, γ_-^+, γ_-^- satisfy

$$\gamma_+^{\pm}(1) = (1, 0) \cdot \gamma_+^{\pm}(0), \quad \gamma_-^{\pm}(1) = (0, 1) \cdot \gamma_-^{\pm}(0),$$

see Figure 5. On the other hand,

$$\big(\langle [p(\gamma_+^{\pm})], \gamma_2 \rangle, -\langle [p(\gamma_+^{\pm})], \gamma_1 \rangle\big) = (1, 0) \text{ and } \big(\langle [p(\gamma_-^{\pm})], \gamma_2 \rangle, -\langle [p(\gamma_-^{\pm})], \gamma_1 \rangle\big) = (0, 1),$$

which completes the proof. $\square$

3. Geometric step of the proof of Theorem 1.2

The proof of Theorem 1.2 can be divided into two parts. The first part relies on Teichmüller dynamics, the Kontsevich-Zorich cocycle and a bounded deviation phenomenon. The corresponding statement, Theorem 3.1 below, ensures the existence (for a.e. Λ) of a non-trivial homology class $\xi \in H_1(M(\Lambda, R), \mathbb{R})$ such that, roughly speaking, the intersection number of ξ with arbitrary vertical orbit segment on $M(\Lambda, R)$ is uniformly bounded. This is technically the most involved part, so we postpone the proof of Theorem 3.1 together with the necessary background until Section 4.

The second part, Theorem 3.2 below, is geometric. We use the homology class $\xi \in H_1(M(\Lambda, R), \mathbb{R})$ to identify the direction of the bands trapping vertical light rays in $F(\Lambda, R)$. More precisely, we show that this direction is given by the vector

$$\bar{v}(\Lambda, \xi) := \langle \gamma_2, \xi \rangle \gamma_+ - \langle \gamma_1, \xi \rangle \gamma_- \in \mathbb{R}^2,$$

for vectors γ_+, γ_- and homology classes γ_1, γ_2 defined in Section 2.

In order to formulate Theorem 3.1 and Theorem 3.2 we need auxiliary notation.

Notation. Let $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ be a $\mathbb{Z}^d$-cover of a compact translation surface (M, ω). Denote by M^+_ω the set of points $x \in M$ such that the positive semi-orbit $(\varphi^v_t(x))_{t \geq 0}$ on (M, ω) is well defined. Let $D \subset \widetilde{M}_\gamma$ be a bounded fundamental domain of the cover such that the interior of D is path-connected and the boundary of D is a finite union of intervals. For every $x \in M^+_\omega$ and $t > 0$ define the element $\sigma^\omega_t(x) \in H_1(M, \mathbb{Z})$ as the homology class of the loop formed by the segment of the vertical orbit of x from x to $\varphi^v_t(x)$ closed up by the shortest curve joining $\varphi^v_t(x)$ with x that does not cross $p^{-1}(\partial D)$.

Following (5), we denote fundamental domains for surfaces $M(\Lambda, R)$ and their covers $\widetilde{M}(\Lambda, R)$ by D.

Theorem 3.1. *Let ω be the Abelian differential on M determining the translation structure on $M(\Lambda, R)$. Then for every $R > 0$ and $\mu_{\mathscr{L}}$-a.e. lattice $\Lambda \in \mathscr{L}$ there exists $0 \neq \xi \in \mathbb{R}\gamma_1 + \mathbb{R}\gamma_2 \subset H_1(M, \mathbb{R})$ and $C > 0$ such that*

$$|\langle \sigma^\omega_t(x), \xi \rangle| \leq C \text{ for every } x \in M^+_\omega \text{ and } t > 0.$$

Theorem 3.2. *Suppose that $R > 0$ and $\Lambda \in \mathscr{L}$ satisfy* (1) *and $M(\Lambda, R) = (M, \omega)$. Further assume that there is a non-zero homology class $\xi \in \mathbb{R}\gamma_1 + \mathbb{R}\gamma_2$ and $C > 0$ such that*

$$|\langle \sigma^\omega_t(x), \xi \rangle| \leq C \text{ for every } x \in M^+_\omega \text{ and } t > 0.$$

If the surface $M(\Lambda, R)$ has no vertical saddle connection, i.e. there is no vertical orbit segment that connect singular points, then there exists $\overline{C} > 0$ such that every vertical orbit on $F(\Lambda, R)$ is trapped in an infinite band in direction $\bar{v}(\Lambda, \xi)$. Furthermore the width of that band is bounded by $\overline{C}$.

Since condition (1) is satisfied for every $R > 0$ and $\mu_{\mathscr{L}}$-a.e. lattice $\Lambda \in \mathscr{L}$ and $M(\Lambda, R)$ has no vertical saddle connection (see Remark 5), Theorem 1.2 is an obvious consequence of the above two theorems.

In the remainder of this section we will prove Theorem 3.2. The proof will be preceded by a series of useful observations. We shall postpone the proof of Theorem 3.1 until Section 4.

Let (M, ω) be a compact translation surface and let $\mathrm{m}(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$ be its $\mathbb{Z}^d$-cover given by $\gamma = (\gamma_1, \ldots, \gamma_d) \in H_1(M, \mathbb{Z})^d$. Denote by $(\widetilde{\varphi}^v_t)_{t \in \mathbb{R}}$ the vertical flow on the $\mathbb{Z}^d$-cover $(\widetilde{M}_\gamma, \widetilde{\omega}_\gamma)$. Let us consider the map $\bar{m} : \widetilde{M}_\gamma \to \mathbb{Z}^d$ such that $\bar{m} = \bar{m}(\widetilde{x}) \in \mathbb{Z}^d$ is the unique element with $\widetilde{x} \in \bar{m} \cdot D$.

Lemma 3.3. *Let $\widetilde{x} \in \widetilde{M}_\gamma$ and $x = p(\widetilde{x}) \in M^+_\omega$. Let $\xi \in H_1(M, \mathbb{R})$ be an element such that $\xi = \sum_{i=1}^d a_i\gamma_i$. Then for every $t > 0$ we have*

$$(a_1, \dots, a_d) \cdot (\bar{m}(\widetilde{\varphi}^v_t \widetilde{x}) - \bar{m}(\widetilde{x})) = \langle \sigma^\omega_t(x), \xi\rangle.$$

Proof. Let $\bar{m} = \bar{m}(\widetilde{\varphi}^v_t \widetilde{x}) - \bar{m}(\widetilde{x})$. Then both $\widetilde{\varphi}^v_t \widetilde{x},\ \bar{m} \cdot \widetilde{x} \in (\bar{m}(\widetilde{x}) + \bar{m}) \cdot D$. Let us consider the curve $\sigma_t(\widetilde{x})$ in $\widetilde{M}_\gamma$ which is formed by the segment of the vertical orbit of $\widetilde{x}$ from $\widetilde{x}$ to $\widetilde{\varphi}^v_t(\widetilde{x})$ together with the shortest curve in $(\bar{m}(\widetilde{x}) + \bar{m}) \cdot D$ joining $\widetilde{\varphi}^v_t(\widetilde{x})$ with $\bar{m} \cdot \widetilde{x}$. By definition, $\sigma^\omega_t(x) = [p \circ \sigma_t(\widetilde{x})]$. Since $(\gamma_1, \dots, \gamma_d)$ determines the cover, it follows that the beginning $\widetilde{x}$ and the end $\bar{m} \cdot \widetilde{x}$ of the curve $\sigma_t(\widetilde{x})$ satisfy

$$\bar{m} \cdot \widetilde{x} = (\langle \sigma^\omega_t(x), \gamma_1\rangle, \dots, \langle \sigma^\omega_t(x), \gamma_d\rangle) \cdot \widetilde{x}.$$

Since the $\mathbb{Z}^d$-action on $\widetilde{M}_\gamma$ is free, it follows that

$$\bar{m} = (\langle \sigma^\omega_t(x), \gamma_1\rangle, \dots, \langle \sigma^\omega_t(x), \gamma_d\rangle).$$

Consequently,

$$(a_1, \dots, a_d) \cdot \bar{m} = \langle \sigma^\omega_t(x), \sum_{i=1}^d a_i\gamma_i\rangle = \langle \sigma^\omega_t(x), \xi\rangle.$$

□

Let us consider the functions $\widetilde{m} : \widetilde{M}(\Lambda, R) \to \mathbb{Z}^2$ and $\widehat{m} : F(\Lambda, R) \to \mathbb{Z}^2$ such that $\widetilde{m}(\widetilde{x}), \widehat{m}(\widehat{x}) \in \mathbb{Z}^2$ are the unique elements with $\widetilde{x} \in \widetilde{m}(\widetilde{x}) \cdot D$ and $\widehat{x} \in \widehat{m}(\widehat{x}) \cdot P(\gamma_+, \gamma_-)$ for all $\widetilde{x} \in \widetilde{M}(\Lambda, R)$ and $\widehat{x} \in F(\Lambda, R)$. In view of (5) and (6), we have

$$\widehat{m}(\zeta\widetilde{x}) = \widetilde{m}(\widetilde{x}) \quad \text{for every} \quad \widetilde{x} \in \widetilde{M}(\Lambda, R) \tag{7}$$

and, by definition, for every $\widehat{x} \in F(\Lambda, R)$ we have

$$\widehat{x} = \widehat{m}_1(\widehat{x})\gamma_+ + \widehat{m}_2(\widehat{x})\gamma_- + \gamma_+ y_1 + \gamma_- y_2 \quad \text{for some } y_1, y_2 \in [-1/2, 1/2). \tag{8}$$

Remark 3. For every $\widehat{x} \in F(\Lambda, R)$ let $\mathcal{O}(\widehat{x})$ be the light ray orbit passing through $\widehat{x}$. If $\mathcal{O}(\widehat{x})$ is a *regular orbit*, i.e. $\mathcal{O}(\widehat{x})$ does not pass through the ends of any slit in $F(\Lambda, R)$ then, by the definition of the light ray flow $(\widetilde{\varphi}_t)_{t\in\mathbb{R}}$ on $\widetilde{M}(\Lambda, R)$, we have

$$\mathcal{O}(\widehat{x}) = \zeta\{\widetilde{\varphi}_t\widetilde{x} : t \in \mathbb{R}\} \text{ for any } \widetilde{x} \in \widetilde{M}(\Lambda, R) \text{ with } \zeta\widetilde{x} = \widehat{x}. \tag{9}$$

Now suppose that an orbit $\mathcal{O}(\widehat{x})$ passes through the ends of slits once and let $\widehat{x}$ be such end. For small $\varepsilon > 0$ let $\widehat{x}_{-\varepsilon} := \widehat{x} - (0, \varepsilon) \in F(\Lambda, R)$ and $\widehat{x}_\varepsilon := \widehat{x} + (0, \varepsilon) \in F(\Lambda, R)$. Next choose $\widetilde{x}_{-\varepsilon}, \widetilde{x}_\varepsilon \in F_+(\Lambda, R) \subset M(\Lambda, R)$ such that $\zeta\widetilde{x}_{-\varepsilon} = \widehat{x}_{-\varepsilon}$ and $\zeta\widetilde{x}_\varepsilon = \widehat{x}_\varepsilon$. Then

$$\mathcal{O}(\widehat{x}) = \bigcup_{\varepsilon>0} \zeta\{\widetilde{\varphi}_t\widetilde{x}_\varepsilon : t \geq 0\} \cup \zeta\{\widetilde{\varphi}_{-t}\widetilde{x}_{-\varepsilon} : t \geq 0\} \cup \{\widehat{x}\}. \tag{10}$$

Let us consider the map $\rho : \widetilde{M}(\Lambda, R) \to \widetilde{M}(\Lambda, R)$ defined by

$$\rho(\widetilde{x}) = \begin{cases} r_\pi \circ \zeta_-^{-1} \circ \zeta_+(\widetilde{x}) & \text{if } \widetilde{x} \in F_+(\Lambda, R), \\ \zeta_+^{-1} \circ \zeta_- \circ r_\pi^{-1}(\widetilde{x}) & \text{if } \widetilde{x} \in r_\pi F_-(\Lambda, R). \end{cases}$$

Lemma 3.4. *The map $\rho : \widetilde{M}(\Lambda, R) \to \widetilde{M}(\Lambda, R)$ is an involution with the properties*

$$\zeta \circ \rho = \zeta \quad \textit{and} \quad \rho^*(\omega) = -\omega. \tag{11}$$

Proof. By definition, ρ is an involution that maps $F_+(\Lambda, R)$ on $r_\pi F_-(\Lambda, R)$ and $r_\pi F_-(\Lambda, R)$ on $F_+(\Lambda, R)$. Comparing this with the definition of ζ (see (3)) immediately gives $\zeta \circ \rho = \zeta$.

Recall that the form ω is given by dz on both $F_+(\Lambda, R)$ and $r_\pi F_-(\Lambda, R)$. The map $\zeta_-^{-1} \circ \zeta_+ : F_+(\Lambda, R) \to F_-(\Lambda, R)$ is a bijection that sends dz to dz. Moreover, the rotation $r_\pi : F_-(\Lambda, R) \to r_\pi F_-(\Lambda, R)$ sends dz to $-dz$. Since ρ is the composition (on pieces) the above maps or their inverses, it follows that $\rho^*(\omega) = -\omega$. □

Note that, by the definition of directional flows, $\rho^*(\omega) = -\omega$ immediately implies

$$\rho \circ \widetilde{\varphi}_t^v(\widetilde{x}) = \widetilde{\varphi}_{-t}^v \circ \rho(\widetilde{x}) \tag{12}$$

for all $\widetilde{x} \in \widetilde{M}(\Lambda, R)$ and $t \in \mathbb{R}$ for which both sides of (12) are well defined.

Proof of Theorem 3.2. Let $\mathcal{O}$ be a trajectory of light in $F(\Lambda, R)$. Since $M(\Lambda, R)$ has no vertical saddle connection, $\mathcal{O}$ passes through the ends of a slit in $F(\Lambda, R)$ at most once.

Suppose first that $\mathcal{O} = \mathcal{O}(\widehat{x})$ is a regular orbit for some $\widehat{x} \in F(\Lambda, R)$. Let $\widetilde{x} \in M(\Lambda, R)$ be such that $\zeta\widetilde{x} = \widehat{x} \in \mathcal{O}$. In view of (9), (12) and (11),

$$\mathcal{O}(\widehat{x}) = \zeta\{\widetilde{\varphi}_t^v\widetilde{x} : t \in \mathbb{R}\} = \zeta\{\widetilde{\varphi}_t^v\widetilde{x} : t \geq 0\} \cup \zeta\{\widetilde{\varphi}_t^v\rho\widetilde{x} : t \geq 0\}$$

and $p(\widetilde{x}), p(\rho\widetilde{x}) \in M_\omega^+$.

As $\xi \in \mathbb{R}\gamma_1 + \mathbb{R}\gamma_2$ and $\langle\gamma_1, \gamma_2\rangle = 2$, we have $\xi = a\gamma_2 - b\gamma_1$ with

$$a = \frac{\langle\gamma_1, \xi\rangle}{\langle\gamma_1, \gamma_2\rangle} = \frac{1}{2}\langle\gamma_1, \xi\rangle \quad \text{and} \quad b = -\frac{\langle\gamma_2, \xi\rangle}{\langle\gamma_2, \gamma_1\rangle} = \frac{1}{2}\langle\gamma_2, \xi\rangle.$$

Since the pair $(\gamma_2, -\gamma_1)$ defines the cover, by Lemma 3.3,

$$\big|(a, b) \cdot (\widetilde{m}(\widetilde{\varphi}_t^v\widetilde{x}) - \widetilde{m}(\widetilde{x}))\big| = |\langle\sigma_t^\omega(p(\widetilde{x})), \xi\rangle| \leq C \quad \text{for} \quad t \geq 0,$$

moreover, by (7), (11) and (12), for $t < 0$ we have

$$\begin{aligned}\big|(a, b) \cdot (\widetilde{m}(\widetilde{\varphi}_t^v\widetilde{x}) - \widetilde{m}(\widetilde{x}))\big| &= \big|(a, b) \cdot (\widetilde{m}(\widetilde{\varphi}_{-t}^v\rho\widetilde{x}) - \widetilde{m}(\rho\widetilde{x}))\big| \\ &= |\langle\sigma_{-t}^\omega(p(\rho\widetilde{x})), \xi\rangle| \leq C.\end{aligned}$$

For every $t \in \mathbb{R}$ let $\widehat{x}_t = \zeta\big(\widetilde{\varphi}_t^v\widetilde{x}\big)$ and

$$\bar{m}^t := \widetilde{m}(\widetilde{\varphi}_t^v\widetilde{x}) - \widetilde{m}(\widetilde{x}) = \widehat{m}(\widehat{x}_t) - \widehat{m}(\widehat{x}),$$

see (7). By (8), this yields

$$\widehat{x}_t = \widehat{x} + \bar{m}_1^t\gamma_+ + \bar{m}_2^t\gamma_- + \gamma_+y_1 + \gamma_-y_2$$

with $y_1, y_2 \in [-1, 1]$. Therefore

$$\widehat{x}_t - \widehat{x} - \frac{(b, -a) \cdot \bar{m}^t}{a^2 + b^2}(b\gamma_+ - a\gamma_-) = \frac{(a, b) \cdot \bar{m}^t}{a^2 + b^2}(a\gamma_+ + b\gamma_-) + \gamma_+y_1 + \gamma_-y_2.$$

Since $|(a, b) \cdot \bar{m}^t| \leq C$ for every $t \in \mathbb{R}$ and

$$b\gamma_+ - a\gamma_- = \frac{1}{\langle\gamma_1, \gamma_2\rangle}(\langle\gamma_2, \xi\rangle\gamma_+ - \langle\gamma_1, \xi\rangle\gamma_-) = \frac{\bar{v}(\Lambda, \xi)}{\langle\gamma_1, \gamma_2\rangle},$$

it follows that

$$\operatorname{dist}\big(\widehat{x}_t - \widehat{x}, \mathbb{R}\,\bar{v}(\Lambda, \xi)\big) \leq \overline{C} := \Big(\frac{C}{\sqrt{a^2 + b^2}} + 1\Big)(\|\gamma_+\| + \|\gamma_-\|).$$

Finally suppose that $\mathcal{O} = \mathcal{O}(\widehat{x})$ is not regular and $\widehat{x}$ is the end of a slit. Then, in view of (10), (12) and (11),

$$\mathcal{O} = \bigcup_{\varepsilon > 0} \zeta\{\widetilde{\varphi}_t^v\widetilde{x}_\varepsilon : t \geq 0\} \cup \zeta\{\widetilde{\varphi}_t^v\rho\widetilde{x}_{-\varepsilon} : t \geq 0\} \cup \{\widehat{x}\}$$

and $p(\widetilde{x}_{-\varepsilon}), p(\rho\widetilde{x}_{-\varepsilon}) \in M_\omega^+$. Now the rest of the proof runs as in the regular case. $\square$

4. Moduli space, Teichmüller flow and Kontsevich-Zorich cocycle

In this section we give a brief overview of the Teichmüller flow and the Kontsevich-Zorich cocycle. For further background material we refer the reader to [3, 4, 13, 16].

Given a connected compact orientable surface M denote by $\mathrm{Diff}^+(M)$ the group of orientation-preserving homeomorphisms of M. We will denote by $\mathcal{M}(M)$ ($\mathcal{M}_a(M)$) the *moduli space* of (area $a>0$) Abelian differentials, that is the space of orbits of the natural action of $\mathrm{Diff}^+(M)$ on the space of (area $a>0$) Abelian differentials on M. Nevertheless $\mathcal{M}_a(M)$ can be identified with $\mathcal{M}_1(M)$, by rescaling Abelian differentials with the factor $1/\sqrt{a}$.

The group $SL(2,\mathbb{R})$ acts naturally on the space of Abelian differentials on M and $\mathcal{M}(M)$ as follows: given a translation structure ω, consider the charts given by local primitives of the holomorphic 1-form. The new charts defined by postcomposition of these charts with an element of $SL(2,\mathbb{R})$ define a new complex structure and a new differential which is holomorphic with respect to this new complex structure, thus a new translation structure. We denote by $g\cdot\omega$ the translation structure on M obtained acting by $g\in SL(2,\mathbb{R})$ on a translation structure ω on M. Since $\mathcal{M}_a(M)$ is $SL(2,\mathbb{R})$-invariant, we restrict the action $SL(2,\mathbb{R})$ to $\mathcal{M}_a(M)$.

Using the Iwasawa NAK decomposition, every element of $SL(2,\mathbb{R})$ has a unique decomposition $h_s g_t r_\theta$, where

$$h_s=\begin{pmatrix} 1 & 0\\ s & 1\end{pmatrix},\quad g_t=\begin{pmatrix} e^t & 0\\ 0 & e^{-t}\end{pmatrix},\quad r_\theta=\begin{pmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.$$

Remark 4. Since the action of $h_s g_t$ rescales the vertical vector $(0,1)$ by the factor e^{-t}, the vertical flow on $(M, h_s g_t\cdot\omega)$ coincides with the linear time change by the factor e^{-t} of the vertical flow on (M,ω). It follows that

$$\sigma_T^{h_s g_t\cdot\omega}(x)=\sigma^{\omega}_{e^tT}(x). \tag{13}$$

The restriction of $SL(2,\mathbb{R})$ on $\mathcal{M}_1(M)$ to the diagonal subgroup $(g_t)_{t\in\mathbb{R}}$ is called the *Teichmüller flow* and we will denote this flow by $(g_t)_{t\in\mathbb{R}}$.

Let M be a surface of genus g and let m be the number of zeros of ω. If α_i, $1\le i\le m$ are degrees of all zeros, one has $2g-2=\sum_{i=1}^m\alpha_i$. Let us denote by $\alpha=(\alpha_1,\dots,\alpha_m)$ and $\mathcal{M}(\alpha)$ the *stratum* consisting of all (M,ω) such that ω has m zeros of degrees $\alpha_1,\dots,\alpha_m$. Then the normalized stratum $\mathcal{M}_1(\alpha)=\mathcal{M}(\alpha)\cap\mathcal{M}_1(M)$ is also $SL(2,\mathbb{R})$-invariant.

The *Kontsevich-Zorich cocycle* $(G_t^{KZ})_{t\in\mathbb{R}}$ is the quotient of the trivial action $(g_t\times Id)_{t\in\mathbb{R}}$ on the product of the space of area one Abelian differentials on M with $H_1(M,\mathbb{R})$ by the action of the group $\mathrm{Diff}^+(M)$. Elements of $\mathrm{Diff}^+(M)$ act on the fiber $H_1(M,\mathbb{R})$ by induced maps. The cocycle $(G_t^{KZ})_{t\in\mathbb{R}}$ acts on the homology vector bundle $\mathcal{H}_1(M,\mathbb{R})$ over the Teichmüller flow $(g_t)_{t\in\mathbb{R}}$ on the moduli space $\mathcal{M}_1(M)$.

Clearly the fibers of the bundle $\mathcal{H}_1(M,\mathbb{R})$ can be identified with $H_1(M,\mathbb{R})$. The algebraic intersection number furnishes $H_1(M,\mathbb{R})$ with a symplectic structure. This symplectic structure is invariant under the action of the mapping-class group and hence invariant under the action of $SL(2,\mathbb{R})$.

The standard definition of the KZ-cocycle is based on the cohomological bundle $\mathcal{H}^1(M,\mathbb{R})$. Each fiber of this bundle (identified with $H^1(M,\mathbb{R})$) is endowed with a natural norm, called the *Hodge norm*, see for example [4] for definition. The identification of the homological and cohomological bundle and the corresponding KZ-cocycles is established by Poincaré duality $\mathcal{P}: H_1(M,\mathbb{R})\to H^1(M,\mathbb{R})$. Via Poincaré duality, the Hodge norms induce norms on the fibers of $\mathcal{H}_1(M,\mathbb{R})$. The norm on the fiber $H_1(M,\mathbb{R})$ over $\omega\in\mathcal{M}(M)$ is denoted by $\|\cdot\|_\omega$ and will be called *Hodge norm* as well.

Let ν be a probability measure on $\mathcal{M}_1(M)$ which is $(g_t)_{t\in\mathbb{R}}$-invariant and ergodic. Suppose that $\mathcal{M}_\nu \subset \mathcal{M}_1(M)$ is a $(g_t)_{t\in\mathbb{R}}$-invariant set with full ν-measure and $V \subset H_1(M,\mathbb{R})$ is a symplectic subspace, i.e. the symplectic form restricted to V is non-degenerate. Moreover, assume that V is invariant for the action induced from $(g_t)_{t\in\mathbb{R}}$-action on $\mathcal{M}_\nu$. Then V defines a subbundle, denoted by $\mathcal{V}$, of the bundle $\mathcal{H}_1(M,\mathbb{R})$ over $\mathcal{M}_\nu$ for which the fibers are identified with V.

Let us consider the KZ-cocycle $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ restricted to V. Since the measure ν is ergodic, by Oseledets' theorem, there exists Lyapunov exponents of $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ with respect to the measure ν. As the action of the Kontsevich-Zorich cocycle is symplectic, its Lyapunov exponents with respect to the measure ν are:

$$\lambda_1^{\mathcal{V}} > \lambda_2^{\mathcal{V}} > \ldots > \lambda_s^{\mathcal{V}} \geq -\lambda_s^{\mathcal{V}} > \ldots > -\lambda_2^{\mathcal{V}} > -\lambda_1^{\mathcal{V}}$$

and for ν-a.e. $\omega \in \mathcal{M}_\nu$ there is a splitting $V = \bigoplus_{i=-s}^{s} V_i(\omega)$ (if $\lambda_s^{\mathcal{V}} = 0$ then $V_{-s}(\omega) = V_s(\omega)$) such that for any $\xi \in V_i(\omega)$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log \|\xi\|_{g_t\omega} = \lambda_i^{\mathcal{V}}.$$

It follows that V has a direct splitting

$$V = E_\omega^+ \oplus E_\omega^0 \oplus E_\omega^- \quad \text{ for } \nu\text{-a.e. } \omega \in \mathcal{M}_\nu,$$

into unstable, central and stable subspaces

$$\begin{aligned}
E_\omega^+ &= \Big\{\xi \in V : \lim_{t\to+\infty} \frac{1}{t} \log \|\xi\|_{g_{-t}\omega} < 0\Big\},\\
E_\omega^0 &= \Big\{\xi \in V : \lim_{t\to\infty} \frac{1}{t} \log \|\xi\|_{g_t\omega} = 0\Big\},\\
E_\omega^- &= \Big\{\xi \in V : \lim_{t\to+\infty} \frac{1}{t} \log \|\xi\|_{g_t\omega} < 0\Big\}.
\end{aligned}$$

The dimension of the stable and unstable subspace is equal to the number of positive Lyapunov exponents of $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$.

The following theorem, crucial to the proof of Theorem 3.1, is based on the phenomenon of bounded deviation discovered by Zorich in his seminal papers [14] and [15]. Its proof can be found in [2] (see Theorem 2) and [5] (see Theorem 4.2 - in the cohomological setting).

Theorem 4.1. *For ν-a.e. $\omega \in \mathcal{M}_\nu$ there exists $C > 0$ such that for every $\xi \in E_\omega^-$, $x \in M_\omega^+$ and $t > 0$ we have $|\langle \sigma_t^\omega(x), \xi\rangle| \leq C\|\xi\|_\omega$.*

5. Branched 2-covers of the torus and the proof of Theorem 3.1

Let us consider the standard torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with two different marked points $u_1, u_2 \in \mathbb{T}$. Denote by M the 2-cover of $\mathbb{T}^2$ branched at u_1 and u_2, see Figure 7. Denote by $p : M \to \mathbb{T}^2$ the covering map. Then the deck group consists of id and the involution $\tau : M \to M$ exchanging the points in fibers over $\mathbb{T}^2 \setminus \{u_1, u_2\}$ and fixing the points u_1, u_2. Let us consider the set

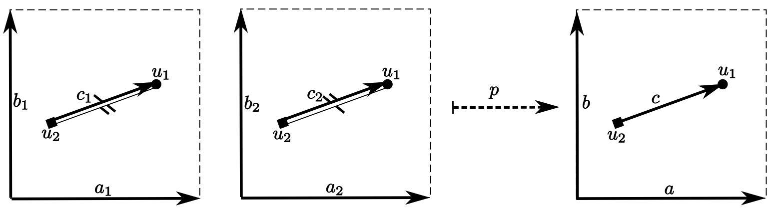


FIGURE 7. Branched 2-cover $p : M \to \mathbb{T}^2$

$$\begin{aligned}\mathcal{M} &= \{(M,\omega) \in \mathcal{M}_2(1,1) : \tau^*\omega = \omega\}\\ &= \{(M,\omega) \in \mathcal{M}_2(1,1) : \omega = p^*\omega_0 \text{ for some } \omega_0 \in \mathcal{M}_1(0,0)\}.\end{aligned}$$

Then $\mathcal{M}$ is an $SL(2,\mathbb{R})$-invariant subset of $\mathcal{M}_2(1,1)$ which is a 2-cover of the stratum $\mathcal{M}_1(0,0)$. Therefore $\mathcal{M}$ has a natural $SL(2,\mathbb{R})$-invariant $(g_t)_{t\in\mathbb{R}}$-ergodic measure $\nu_{\mathcal{M}}$ which is the pullback by the covering map of the canonical measure on the stratum $\mathcal{M}_1(0,0)$.

Let us consider the orthogonal (symplectic) decomposition $H_1(M,\mathbb{R}) = V \oplus V^\perp$ with

$$V = \{\xi \in H_1(M,\mathbb{R}) : \tau_*\xi = -\xi\} \text{ and } V^\perp = \{\xi \in H_1(M,\mathbb{R}) : \tau_*\xi = \xi\}.$$

Then V and $V^\perp$ are 2-dimensional symplectic subspaces and $\gamma_1 := a_1 - a_2$, $\gamma_2 := b_1 - b_2 \in H_1(M,\mathbb{Z})$ establish a basis of V. Since $V = \ker p_*$ and p is $SL(2,\mathbb{R})$-equivariant, V is invariant under the induced $SL(2,\mathbb{R})$-action. Therefore, V defines a subbundle over $\mathcal{M}$ and we can consider the restricted KZ cocycle $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ on $\mathcal{V}$. As it was shown in [1] by Bainbridge, the Lyapunov exponents of every ergodic $SL(2,\mathbb{R})$-invariant measure on $\mathcal{M}_2(1,1)$ are equal to 1, $1/2$, $-1/2$, and -1. It follows that the Lyapunov exponents of $(G_t^{\mathcal{V}})_{t\in\mathbb{R}}$ are $1/2$ and $-1/2$. Therefore, for $\nu_{\mathcal{M}}$-a.e. $\omega \in \mathcal{M}$ the stable subspace $E_\omega^- \subset V$ is one-dimensional (so non-trivial). We now apply Theorem 3.2 to the measure $\nu_{\mathcal{M}}$ to obtain the following result.

Lemma 5.1. *For $\nu_{\mathcal{M}}$-a.e. surface (M,ω)*

(14) *there exist $\xi \in V\setminus\{0\}, C > 0$ such that $|\langle\sigma_t^\omega(x), \xi\rangle| \le C$ for all $x \in M_\omega^+, t > 0$.*

Proof of Theorem 3.1. We begin the proof by describing a local product structure on $\mathcal{M}$ that will help us to deduce Theorem 3.1 directly from Lemma 5.1.

Since $\mathcal{M} \subset \mathcal{M}_2(1,1)$ is a 2-cover of the stratum $\mathcal{M}_1(0,0)$, it is a 5 dimensional manifold. Each element (M,ω) of $\mathcal{M}$ is a union of two identical tori glued along a slit, see Figure 8.

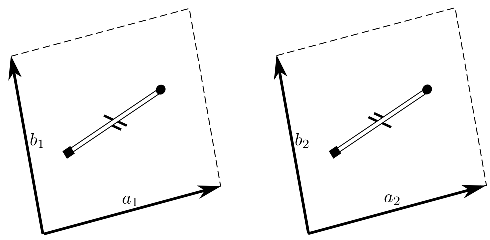


FIGURE 8. Typical surface in $\mathcal{M}$

For fixed $R > 0$ let $\Upsilon : SL(2,\mathbb{R})/SL(2,\mathbb{Z}) \times \mathbb{R}^2 \to \mathcal{M}$ be the map

$$\Upsilon(\Lambda, t, s) = (h_s g_t) \cdot M(\Lambda, R).$$

This map is a local diffeomorphism whose image consists of surfaces in $\mathcal{M}$ whose slits are not vertical. Indeed, local inverses of Υ are given as follows. To start represent every surface (M,ω) in $\mathcal{M}$ as the union of two identical tori glued along a slit, see Figure 8. Assume that the slit is not vertical. Then transform (M,ω) by h_{-s} so that the slit of the resulting surface $(M, h_{-s}\cdot\omega)$ is horizontal. Next transform $(M, h_{-s}\cdot\omega)$ by g_{-t} so that the length of the horizontal slit for the resulting surface $(M, (g_{-t}h_{-s})\cdot\omega)$ is $2R$. Therefore, $(M, (g_{-t}h_{-s})\cdot\omega) = M(\Lambda, R)$ for some $\Lambda \in \mathscr{L}$, so $(M,\omega) = \Upsilon(\Lambda, s, t)$.

Since $\nu_{\mathcal{M}}$ is a smooth measure on $\mathcal{M}$ and Υ is a local diffeomorphism, the image by Υ of the product measure $\mu_{\mathscr{L}} \otimes Leb_{\mathbb{R}^2}$ on $\mathscr{L} \times \mathbb{R}^2$ is equivalent to the measure $\nu_{\mathcal{M}}$.

The local product structure on $\mathcal{M}$ arising from Υ helps to deduce Theorem 3.1 form Lemma 5.1. Indeed, suppose, contrary to the claim of Theorem 3.1, that there exists a measurable subset $\mathscr{L}_0 \subset \mathscr{L}$ of positive $\mu_{\mathscr{L}}$-measure such that for every $\Lambda \in \mathscr{L}_0$ condition (14) does not hold for the surface $M(\Lambda, R)$. In view of (13), condition (14) does not hold for every surface $(h_s g_t) \cdot M(\Lambda, R)$ with $s, t \in \mathbb{R}$. Consequently, for every surface from the set $\Upsilon(\mathscr{L}_0 \times [0,1]^2)$ condition (14) is not valid. Since the set has positive $\nu_{\mathcal{M}}$-measure, one gets a contradiction to Lemma 5.1. $\square$

Remark 5. Note that for every translation surface (M, ω) for a.e. $\theta \in S^1$ the vertical flow on $(M, r_\theta \omega)$ has no saddle connection. It follows from Fubini theorem, that if ν is an $SL(2,\mathbb{R})$-invariant measure on the moduli space $\mathcal{M}_1(M)$ then for ν-a.e. (M, ω) the vertical flow on (M, ω) has no saddle connection. Applying this observation to the measure $\nu_{\mathcal{M}}$ and then proceeding as in the proof of Theorem 3.1, we obtain that for every $R > 0$ and $\mu_{\mathscr{L}}$-a.e. lattice $\Lambda \in \mathscr{L}$ the surface $M(\Lambda, R)$ has no saddle connections.

6. Periodic case and examples

In this section we will show how to determine the direction of confining strips when the surface $M(\Lambda, R)$ is a periodic point of the Teichmüller flow. Under additional assumption (hyperbolicity of a certain matrix) we present a procedure that helps to construct examples of pairs Λ, R for which the direction of confining strips on $L(\Lambda, R)$ can be computed. We conclude this section with a specific example.

Suppose that $M(\Lambda, R) = (M, \omega_0) \in \mathcal{M}$ is a periodic point of the Teichmüller flow with period $t_0 > 0$. Since $g_{t_0}(M, \omega_0) = (M, \omega_0)$ in the moduli space, there exists an affine homeomorphism $\psi : (M, \omega_0) \to (M, \omega_0)$ such that $D\psi = g_{t_0}$. Note that the group of affine homeomorphisms with derivative being the identity consists of id and τ. Since $\psi \circ \tau$ is affine with $D(\psi \circ \tau) = g_{t_0}$, we have that either $\psi \circ \tau = \tau \circ \psi$ or $\psi \circ \tau = \psi$. As ψ is one-to-one, the latter equality is not satisfied, so $\psi \circ \tau = \tau \circ \psi$. Therefore $\psi_* \circ \tau_* = \tau_* \circ \psi_*$, hence ψ_* preserves the subspaces V and $V^\perp$. Since ψ_* restricted to $V \cap H_1(M, \mathbb{Z})$ is an automorphism, any matrix representation of $\psi_* : V \to V$ is an element of $SL(2, \mathbb{Z})$. Suppose that this element is hyperbolic. Then there exist eigenvectors $\xi_s, \xi_u \in V$ such that

$$\psi_* \xi_s = \lambda \xi_s \quad \text{and} \quad \psi_* \xi_u = \lambda^{-1} \xi_u \quad \text{with } |\lambda| < 1.$$

Lemma 6.1. *There exists $C > 0$ such that $|\langle \sigma_T^{\omega_0}(x), \xi_s \rangle| \le C$ for all $x \in M^+_{\omega_0}$ and $T > 0$.*

Proof. Denote by $\mathcal{M}_0$ the Teichmüller flow orbit of (M, ω_0). This is an invariant set with the unique probability invariant measure $\nu_{\mathcal{M}_0}$ which is the image of the Lebesgue measure on the circle $\mathbb{R}/t_0\mathbb{Z}$ via the map $\mathbb{R}/t_0\mathbb{Z} \ni t \mapsto g_t(M, \omega_0) \in \mathcal{M}$. Let us consider the KZ-cocycle $(G_t^{\mathcal{V}})_{t \in \mathbb{R}}$ on $\mathcal{V}$ over the set $\mathcal{M}_0$. The cocycle is completely determined by the map $\psi_* : V \to V$. Since it is hyperbolic we have

$$E^-_\omega = \mathbb{R}\xi_s \quad \text{and} \quad E^+_\omega = \mathbb{R}\xi_u \quad \text{for every} \quad \omega \in \mathcal{M}_0.$$

In view of Theorem 4.1, for almost every $t \in \mathbb{R}/t_0\mathbb{Z}$ there exists $C > 0$ such that $|\langle \sigma_T^{g_t\omega_0}(x), \xi_s \rangle| \le C$ for all $x \in M$ and $T \in \mathbb{R}$. By (13), this yields the assertion of the lemma. $\square$

Fix $R > 0$ and consider a lattice $\Lambda \in \mathscr{L}$ satisfying (1). Let us choose a positive basis γ_+, γ_- of Λ satisfying (2). Then there exists a unique $g \in SL(2, \mathbb{R})$ such that $g(\gamma_+) = (1, 0)$ and $g(\gamma_-) = (0, 1)$. Denote by $\vartheta \in S^1$ the direction of the vector $g(0, 1)$ and $u := g(R, 0)$. Then $u \in (-1/2, 1/2)^2 \setminus \{(0, 0)\}$ and $u \wedge \vartheta > 0$.

Let us consider the surface $g \cdot M(\Lambda, R) \in \mathcal{M}$. This translation surface is the union of two copies of the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ with translation structure inherited from the Euclidean plane (as a model torus we will use the square $[-1/2, 1/2)^2$) glued along the linear slit between u and $-u$, see Figure 9. Such surface will be denoted by $M(u)$.

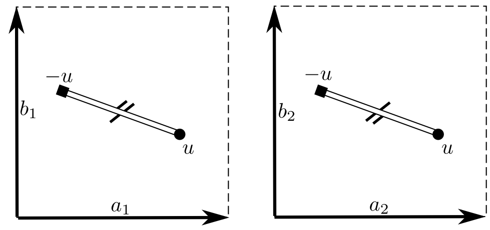


FIGURE 9. Translation surface $M(u)$

Note that the map

$$(\Lambda, \gamma_+, \gamma_-) \mapsto (u, \vartheta) \tag{15}$$

gives a one-to-one correspondence between translation surfaces $M(\Lambda, R)$ with assigned oriented positive basis γ_+, γ_- and translation surfaces $M(u)$ with assigned direction ϑ such that $u \wedge \vartheta > 0$. The inverse of (15) is given by $\eta_{u,\vartheta} \in SL(2,\mathbb{R})$ such that $\eta_{u,\vartheta} u = (R, 0)$ and $\eta_{u,\vartheta}\vartheta = (0, (u \wedge \vartheta)/R)$. Then taking $\gamma_+ = \eta_{u,\vartheta}(1,0)$, $\gamma_- = \eta_{u,\vartheta}(0,1)$ and $\Lambda_{u,\vartheta} = \mathbb{Z}\gamma_+ + \mathbb{Z}\gamma_-$, we get $M(\Lambda_{u,\vartheta}, R) = \eta_{u,\vartheta} M(u)$.

By $\mathscr{U} \subset \mathcal{M}$ we denote the set of translation surfaces $M(u)$ for which

$$u \in \mathbb{T}^2_0 := [-1/2, 1/2) \times [-1/2, 1/2) \setminus \{(0,0), (-1/2, -1/2), (-1/2, 0), (0, -1/2)\}.$$

The set $\mathscr{U}$ is $SL(2,\mathbb{Z})$-invariant and $g \cdot M(u) = M(gu)$, where $gu \in \mathbb{T}^2_0$ is the image of u by the algebraic automorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ defined by the matrix $g \in SL(2,\mathbb{Z})$, see [6].

The $SL(2,\mathbb{Z})$-action on $\mathscr{U}$ defines the induced homology action on $H_1(M, \mathbb{R})$ for which the subspace V is invariant, see also [6]. For every translation surface $M(u)$ we choose canonically the basis $\{e_1, e_2\}$ of V, where $e_1 := a_1 - a_2$ and $e_2 = b_1 - b_2$. Since $\gamma_+ = \eta_{u,\vartheta}(1,0)$ and $\gamma_- = \eta_{u,\vartheta}(0,1)$, we have $\gamma_1 = (\eta_{u,\vartheta})_* e_1$ and $\gamma_2 = (\eta_{u,\vartheta})_* e_2$.

For every $g \in SL(2,\mathbb{Z})$ and $u \in \mathbb{T}^2_0$ denote by $g_*(u)$ the matrix representation in canonical bases of the homology map induced by $g : M(u) \to M(gu)$. Since $g_*(u)$ is well defined up to multiplication by ± 1 (see [6]), it is treated as an element of $PSL(2,\mathbb{Z})$.

Theorem 6.2. *Let $u \in (-1/2, 1/2)^2 \setminus \{(0,0)\}$ be rational. Suppose that $h \in SL(2,\mathbb{Z})$ is a hyperbolic matrix such that $hu = u$ (in $\mathbb{T}^2_0$) and $h_*(u)$ is also hyperbolic. Denote by $\vartheta, \theta \in S^1$ the contracting eigendirections of h and $h_*(u)$ respectively. Then all vertical light rays in $F(\Lambda_{u,\vartheta}, R)$ are trapped in infinite bands in direction $\eta_{u,\vartheta}\theta$.*

Proof. Denote by $t_0 > 0$ the natural logarithm of the largest eigenvalue of h. Since $hu = u$ in $\mathbb{T}^2_0$, there exists an affine automorphism $\phi : M(u) \to M(u)$ such that $D\phi = h$. Let us consider the affine automorphism $\eta_{u,\vartheta} \circ \phi \circ \eta^{-1}_{u,\vartheta} : M(\Lambda, R) \to M(\Lambda, R)$ with $\Lambda := \Lambda_{u,\vartheta}$. Then $D\big(\eta_{u,\vartheta} \circ \phi \circ \eta^{-1}_{u,\vartheta}\big) = \eta_{u,\vartheta} \circ h \circ \eta^{-1}_{u,\vartheta}$ and $(0,1)$ is its contracting eigenvector for the eigenvalue e^{-t_0}. Take an expanding eigenvector of the form $(1, r)$ for some $r \in \mathbb{R}$. Let us consider the surface $(M, \omega_0) = h^T_{-r} M(\Lambda, R)$ (h^T_{-r} is the transpose of h_{-r}) and its affine automorphism

$$\psi := h^T_{-r} \circ \eta_{u,\vartheta} \circ \phi \circ \eta^{-1}_{u,\vartheta} \circ h^T_r.$$

Then $(0,1)$ is a contracting eigenvector of $D\psi$ and $(1,0)$ is an expanding one. Therefore, $D\psi = g_{t_0}$. It follows that (M,ω_0) is a periodic element for the Teichmüller flow with period t_0. Since ψ is conjugate to ϕ by $h^T_{-r}\circ\eta_{u,\vartheta}$ and the subspace $V\subset H_1(M,\mathbb{R})$ is $SL(2,\mathbb{R})$-invariant, the induced automorphisms $\phi_*: V\to V$ and $\psi_*: V\to V$ are isomorphic via the map $(h^T_{-r}\circ\eta_{u,\vartheta})_*: V\to V$. As $h_*(u)$ is the matrix representation of $\phi_*: V\to V$ in the basis e_1,e_2 and $\theta=(\theta_1,\theta_2)$ is a contracting eigenvector for $h_*(u)$, the homology class $\theta_1e_1+\theta_2e_2\in V$ is a contracting eigenvector for $\phi_*: V\to V$. It follows that $\xi_s := (h^T_{-r}\circ\eta_{u,\vartheta})_*(\theta_1e_1+\theta_2e_2)\in V$ is a contracting eigenvector for $\psi_*: V\to V$. In view of Lemma 6.1, there exists $C>0$ such that

$$|\langle\sigma^{\omega_0}_T(x),\xi_s\rangle|\leq C \text{ for all } x\in M^+_{\omega_0} \text{ and } T>0.$$

Let us consider the map $h^T_r:(M,\omega_0)\to M(\Lambda,R)$ and its induced transformation $(h^T_r)_*$ on homologies. Since $(h^T_r)_*\sigma^{\omega_0}_T(x)=\sigma^{h^T_r\omega_0}_T(h^T_rx)$ and $(h^T_r)_*$ preserves algebraic intersection number, we also have

$$|\langle\sigma^{h^T_r\omega_0}_T(x),(h^T_r)_*\xi_s\rangle|\leq C \text{ for all } x\in M^+_{h^T_r\omega_0} \text{ and } T>0$$

with $(M,h^T_r\omega_0)=M(\Lambda,R)$.

Since $(\eta_{u,\vartheta})_*$ sends the basis $\{e_1,e_2\}$ to $\{\gamma_1,\gamma_2\}$, we have $(h^T_r)_*\xi_s=\theta_1\gamma_1+\theta_2\gamma_2$. In view of Theorem 3.2, it follows that there exists $\overline{C}>0$ such that every vertical orbit on $F(\Lambda,R)$ is trapped in an infinite band in the direction of the vector

$$\bar v(\Lambda,(h^T_r)_*\xi_s)=\langle\gamma_2,(h^T_r)_*\xi_s\rangle\gamma_+-\langle\gamma_1,(h^T_r)_*\xi_s\rangle\gamma_-=-2(\theta_1\gamma_++\theta_2\gamma_-)=-2\eta_{u,\vartheta}\theta$$

and whose width is bounded by $\overline{C}$. □

Recall hat the matrices $h^+ := \begin{pmatrix}1&1\\0&1\end{pmatrix}$, $h^- := \begin{pmatrix}1&0\\1&1\end{pmatrix}$ generate $SL(2,\mathbb{Z})$. Therefore, the following result allows us to calculate $g_*(u)$ for every $g\in SL(2,\mathbb{Z})$ and $u\in\mathbb{T}^2_0$. Then using Theorem 6.2, one can construct an explicit example of a lattice Λ and $R>0$ such that vertical light rays in $L(\Lambda,R)$ are trapped in strips, see Example 1.

Proposition 6.3 (see [6]). *Set $S:=\{(x,y)\in\mathbb{T}^2_0: -1/2\leq x+y<1/2\}$. For every $u\in\mathbb{T}^2_0$ we have*

$$h^{\pm}_*(u)=\begin{cases} h^{\pm} & \text{if} \quad u\in S,\\ (h^{\pm})^{-1} & \text{if} \quad u\notin S.\end{cases}$$

Example 1. Let $R=1/3$ and let us start from the point $u=(1/3,0)\in\mathbb{T}^2_0$. Take $h=(h^-)^3h^+=\begin{pmatrix}1&1\\3&4\end{pmatrix}$. Then $hu=u$, and using Proposition 6.3, one can compute that $h_*(u)=h^-h^+=\begin{pmatrix}1&1\\1&2\end{pmatrix}$. Therefore, ϑ and θ are directions of contracting eigenvectors $\big(-\frac{3+\sqrt{21}}{6},1\big)$ and $\big(-\frac{1+\sqrt{5}}{2},1\big)$ resp. Then $\eta_{u,\vartheta}=\begin{pmatrix}1&(3+\sqrt{21})/6\\0&1\end{pmatrix}$ and

$$\eta_{u,\vartheta}\Big(-\frac{1+\sqrt{5}}{2},1\Big)=\Big(\frac{\sqrt{21}-3\sqrt{5}}{6},1\Big).$$

It follows that for the lattice

$$\Lambda=\eta_{u,\vartheta}\mathbb{Z}^2=(1,0)\,\mathbb{Z}+\Big(\frac{3+\sqrt{21}}{6},1\Big)\,\mathbb{Z}$$

every vertical light ray in $L(\Lambda,1/3)$ is trapped in a band with slope $-\frac{\sqrt{21}+3\sqrt{5}}{4}$.

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