Circle extensions of $\mathbb{Z}^d$-rotations on the d-dimensional torus

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August 16, 2005

Abstract

Let $\boldsymbol{T}$ be an ergodic and free $\mathbb{Z}^d$-rotation on the d-dimensional torus $\mathbb{T}^d$ given by

$$\begin{aligned}&\boldsymbol{T}_{(m_1,...,m_d)}(z_1,...,z_d)\\ &\qquad = (e^{2\pi i(\alpha_{11}m_1+...+\alpha_{1d}m_d)}z_1,...,e^{2\pi i(\alpha_{d1}m_1+...+\alpha_{dd}m_d)}z_d),\end{aligned}$$

where $(m_1,...,m_d)\in\mathbb{Z}^d$, $(z_1,...,z_d)\in\mathbb{T}^d$ and $[\alpha_{jk}]_{j,k=1...d}\in M_d(\mathbb{R})$. For a continuous circle cocycle $\phi:\mathbb{Z}^d\times\mathbb{T}^d\to\mathbb{T}$ ($\phi_{\boldsymbol{m}+\boldsymbol{n}}(\boldsymbol{z})=\phi_{\boldsymbol{m}}(\boldsymbol{T}_{\boldsymbol{n}}\boldsymbol{z})\phi_{\boldsymbol{n}}(\boldsymbol{z})$ for any $\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^d$), we define the winding matrix $W(\phi)$ of a cocycle ϕ, which is a generalization of the topological degree. We study spectral properties of extensions given by

$$\boldsymbol{T}_\phi:\mathbb{Z}^d\times\mathbb{T}^d\times\mathbb{T}\to\mathbb{T}^d\times\mathbb{T},\ (\boldsymbol{T}_\phi)_{\boldsymbol{m}}(\boldsymbol{z},\omega)=(\boldsymbol{T}_{\boldsymbol{m}}\boldsymbol{z},\phi_{\boldsymbol{m}}(\boldsymbol{z})\omega).$$

We show that if ϕ is smooth (for example ϕ is of class C^1) and $\det W(\phi)\neq 0$, then $\boldsymbol{T}_\phi$ is mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. For $d=2$ we show that if ϕ is smooth (for example ϕ is of class C^4), $\det W(\phi)\neq 0$ and $\boldsymbol{T}$ is a $\mathbb{Z}^2$-rotation of finite type, then $\boldsymbol{T}_\phi$ has countable Lebesgue spectrum on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. If rank $W(\phi)=1$, then $\boldsymbol{T}_\phi$ has singular spectrum.

Introduction

Let X be a compact abelian group and let μ be the probability Haar measure of X. Assume that $\mathbb{G}$ is a countable discrete abelian group and $\Phi:\mathbb{G}\to X$ is a group homomorphism. We will call a $\mathbb{G}$–action on X given by

$$\boldsymbol{T}_g x=\Phi(g)x$$

the $\mathbb{G}$–*rotation* on X. The $\mathbb{G}$–rotation $\boldsymbol{T}$ is ergodic and free iff Φ is monomorphic and $\Phi(\mathbb{G})$ is dense in X. Let H be a locally compact abelian group. Throughout this paper H will be the circle or real line.

1991 *Mathematics Subject Classification*: 28D05.

Research partly supported by KBN grant 2 P301 031 07 (1994)

Definition 0.1. By an H–*cocycle* of the $\mathbb{G}$–rotation $\boldsymbol{T}$ we mean a measurable function $\phi : \mathbb{G} \times X \to H$ such that

$$\phi_{g_1 g_2}(x) = \phi_{g_1}(\boldsymbol{T}_{g_2} x)\phi_{g_2}(x)$$

for any $g_1, g_2 \in \mathbb{G}$ and $x \in X$.

We will call ϕ suitably smooth if the function ϕ_g is smooth for any $g \in \mathbb{G}$. Suppose that H is a compact group and let m be the probability Haar measure of H. Given an H–cocycle ϕ consider the $\mathbb{G}$–action $\boldsymbol{T}_\phi : \mathbb{G} \to \mathrm{Aut}(X \times H, \mathcal{B}, \mu \times m)$ given by

$$(\boldsymbol{T}_\phi)_g(x, h) = (\boldsymbol{T}_g x, \phi_g(x)h),$$

where $\mathcal{B}$ is the product σ–algebra of the Boolean σ–algebras and $\mathrm{Aut}(X \times H, \mathcal{B}, \mu \times m)$ is the group of all measure–preserving automorphisms. The $\mathbb{G}$–action $\boldsymbol{T}_\phi$ is called an H–extension of $\boldsymbol{T}$. In this paper we will consider circle extensions of rotations on the torus.

By $\mathbb{T}^d$ ($d \in \mathbb{N}$) we mean the d-dimensional torus $\{(z_1, ..., z_d) \in \mathbb{C}^d; |z_1| = ... = |z_d| = 1\}$. We will also use the additive notation, i.e. we will identify the group $\mathbb{T}^d$ with the group $\mathbb{R}^d/\mathbb{Z}^d$. We will also identify functions on $\mathbb{R}^d/\mathbb{Z}^d$ with the $\mathbb{Z}^d$–periodic functions on $\mathbb{R}^d$ (periodic of period 1 in each coordinates). Let λ^d denote the probability Lebesgue measure on $\mathbb{T}^d$. Let $\Phi : \mathbb{Z}^d \to \mathbb{T}^d$ be a group homomorphism. Then there exists a matrix $\boldsymbol{\alpha} = [\alpha_{jk}]_{j,k=1...d} \in M_d(\mathbb{R})$ such that

$$\Phi(m_1, ..., m_d) = (e^{2\pi i(\alpha_{11}m_1 + ... + \alpha_{1d}m_d)}, ..., e^{2\pi i(\alpha_{d1}m_1 + ... + \alpha_{dd}m_d)}).$$

Consider a $\mathbb{Z}^d$–rotation $\boldsymbol{T}$ on $\mathbb{T}^d$ given by

$$\boldsymbol{T}_{\boldsymbol{m}}\, \boldsymbol{z} = \Phi(\boldsymbol{m})\, \boldsymbol{z} = (e^{2\pi i(\alpha_{11}m_1 + ... + \alpha_{1d}m_d)} z_1, ..., e^{2\pi i(\alpha_{d1}m_1 + ... + \alpha_{dd}m_d)} z_d),$$

where $\boldsymbol{m} = (m_1, ..., m_d) \in \mathbb{Z}^d$ and $\boldsymbol{z} = (z_1, ..., z_d) \in \mathbb{T}^d$.

Lemma 0.1. *$\boldsymbol{T}$ is ergodic iff $\boldsymbol{m}\,\boldsymbol{\alpha} \notin \mathbb{Z}^d$ for any $\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\}$. $\boldsymbol{T}$ is free iff $\boldsymbol{\alpha}\,\boldsymbol{m}^T \notin \mathbb{Z}^d$ for any $\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\}$.* ■

Write $\boldsymbol{T}_j = \boldsymbol{T}_{(0,...,0,\overset{j}{1},0,...,0)}$ for $j = 1, ..., d$. For any function $\psi : \mathbb{T}^d \to \mathbb{T}$, any $n \in \mathbb{Z}$ and $j = 1, ..., d$ set

$$\psi^{(n,j)}(\boldsymbol{z}) = \begin{cases} \psi(\boldsymbol{z})\psi(\boldsymbol{T}_j\, \boldsymbol{z})...\psi(\boldsymbol{T}_j^{n-1}\, \boldsymbol{z}) & \text{if} \quad n > 0 \\ 1 & \text{if} \quad n = 0 \\ (\psi(\boldsymbol{T}_j^{n}\, \boldsymbol{z})\psi(\boldsymbol{T}_j^{n+1}\, \boldsymbol{z})...\psi(\boldsymbol{T}_j^{-1}\, \boldsymbol{z}))^{-1} & \text{if} \quad n < 0. \end{cases}$$

Let $\phi : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}$ be a $\mathbb{T}$–cocycle. Then ϕ can be represented as

$$\phi_{\boldsymbol{m}}(\boldsymbol{z}) = \phi_1^{(m_1,1)}(\boldsymbol{T}_2^{m_2}\, \boldsymbol{T}_3^{m_3} ... \boldsymbol{T}_d^{m_d}\, \boldsymbol{z})\phi_2^{(m_2,2)}(\boldsymbol{T}_3^{m_3} ... \boldsymbol{T}_d^{m_d}\, \boldsymbol{z}) ... \phi_d^{(m_d,d)}(\boldsymbol{z}),$$

where $\phi_j = \phi_{(0,...,0,\overset{j}{1},0,...,0)}$ for $j = 1, ..., d$. Moreover, for any $j, k = 1, ..., d$ we have

$$\phi_j(\boldsymbol{T}_k\, \boldsymbol{z})\phi_j(\boldsymbol{z})^{-1} = \phi_k(\boldsymbol{T}_j\, \boldsymbol{z})\phi_k(\boldsymbol{z})^{-1}. \tag{1}$$

Suppose that ϕ is a continuous cocycle. Then $\phi_1, ..., \phi_d$ can be represented as

$$\begin{aligned}
\phi_1(e^{2\pi i x_1}, ..., e^{2\pi i x_d}) &= e^{2\pi i(h_1(x_1,...,x_d)+w_{11}x_1+...+w_{1d}x_d)} \\
&\cdots \\
\phi_d(e^{2\pi i x_1}, ..., e^{2\pi i x_d}) &= e^{2\pi i(h_d(x_1,...,x_d)+w_{d1}x_1+...+w_{dd}x_d)},
\end{aligned}$$

where $W(\phi) = [w_{jk}]_{j,k=1...d} \in M_d(\mathbb{Z})$ and $h_1, ..., h_d : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ are continuous. In the above representation of ϕ, the matrix $W(\phi)$ is unique, while $h_1, ..., h_d$ are unique up to an additive integer constant. We call the matrix $W(\phi)$ the *winding matrix* of the cocycle ϕ.

For $j = 1, ..., d$ let $T_j : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be a translation given by

$$T_j(x_1, ..., x_d) = (x_1 + \alpha_{1j}, ..., x_d + \alpha_{dj}).$$

Let T be a $\mathbb{Z}^d$–action on $\mathbb{R}^d/\mathbb{Z}^d$ given by $T_{\boldsymbol{m}} = T_1^{m_1} \circ ... \circ T_d^{m_d}$. From (1) we have

$$\exp(2\pi i(h_j(T_k\,\boldsymbol{x}) - h_j(\boldsymbol{x}) + \sum_{l=1}^{d} w_{jl}\alpha_{lk})) = \exp(2\pi i(h_k(T_j\,\boldsymbol{x}) - h_k(\boldsymbol{x}) + \sum_{l=1}^{d} w_{kl}\alpha_{lj})).$$

It follows that

$$h_j(T_k\,\boldsymbol{x}) - h_j(\boldsymbol{x}) - (h_k(T_j\,\boldsymbol{x}) - h_k(\boldsymbol{x})) + (W\boldsymbol{\alpha})_{jk} - (W\boldsymbol{\alpha})_{kj} = d_{jk} \in \mathbb{Z}.$$

Since

$$\int_{\mathbb{T}^d} (h_j(T_k\,\boldsymbol{x}) - h_j(\boldsymbol{x}) - (h_k(T_j\,\boldsymbol{x}) - h_k(\boldsymbol{x})))d\,\boldsymbol{x} = 0,$$

we have

$$h_j(T_k\,\boldsymbol{x}) - h_j(\boldsymbol{x}) = h_k(T_j\,\boldsymbol{x}) - h_k(\boldsymbol{x})$$

for $j, k = 1, ..., d$ and

$$(W\boldsymbol{\alpha}) - (W\boldsymbol{\alpha})^T \in M_d(\mathbb{Z}). \tag{2}$$

For any function $f : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$, any $n \in \mathbb{Z}$ and $j = 1, ..., d$ set

$$f^{(n,j)}(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) + f(T_j\,\boldsymbol{x}) + ... + f(T_j^{n-1}\,\boldsymbol{x}) & \text{if} \quad n > 0 \\ 0 & \text{if} \quad n = 0 \\ -(f(T_j^n\,\boldsymbol{x}) + f(T_j^{n+1}\,\boldsymbol{x}) + ... + f(T_j^{-1}\,\boldsymbol{x})) & \text{if} \quad n < 0. \end{cases}$$

Then $h = h(\phi) : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ given by

$$h_{\boldsymbol{m}}(\boldsymbol{x}) = h_1^{(m_1,1)}(T_2^{m_2}T_3^{m_3}...T_d^{m_d}\,\boldsymbol{x}) + h_2^{(m_2,2)}(T_3^{m_3}...T_d^{m_d}\,\boldsymbol{x}) + ... + h_d^{(m_d,d)}(\boldsymbol{x})$$

is a real cocycle.

In the case $d = 1$, the cocycle ϕ has only one generator ϕ_1 and the winding matrix of ϕ is the topological degree of ϕ_1. Then we have some information on spectral properties of $\boldsymbol{T}_\phi$. It has been proved by Choe in [1] that if ϕ is of class C^2 and $W(\phi) \neq 0$, then $\boldsymbol{T}_\phi$ has countable Lebesgue spectrum on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. The assumptions for ϕ were weakened in [8] to ϕ absolutely continuous and ϕ' of bounded variation to get countable Lebesgue spectrum. In [8] the authors have proved also that if ϕ is absolutely continuous, then $\boldsymbol{T}_\phi$ is mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. In [7] a sufficient condition for countable Lebesgue spectrum is expressed in terms of the Fourier coefficients of ϕ. On the other hand, in [4] the authors have proved that if $W(\phi) = 0$ and ϕ is absolutely continuous, then $\boldsymbol{T}_\phi$ has singular spectrum.

The aim of this paper is to study the spectral properties of cocycles for $d > 1$. We will try to generalize the above results. We show that if ϕ is weakly absolutely continuous and $\det W(\phi) \neq 0$, then $\boldsymbol{T}_\phi$ is mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. For $d = 2$ we show that if we put a stronger assumption on ϕ (for example ϕ of class C^4), and $\boldsymbol{T}$ is a $\mathbb{Z}^2$-rotation of finite type (i.e. $\boldsymbol{T}$ is slowly approximate to rational rotations), then $\boldsymbol{T}_\phi$ has countable Lebesgue spectrum on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. In the case $\det W(\phi) = 0$ we prove that if rank $W(\phi) = 1$ (or rank $W(\phi) = 0$ and there is an $\boldsymbol{m} \in \mathbb{Z}^2 \setminus \{0\}$ such that the automorphism $\boldsymbol{T}_{\boldsymbol{m}}$ is not ergodic) and ϕ is absolutely continuous, then $\boldsymbol{T}_\phi$ has singular spectrum.

1 Notation and facts from spectral theory

Let U be a unitary representation of group $\mathbb{G}$ in a separable Hilbert space $\mathcal{H}$. For any $f \in \mathcal{H}$ we define the *cyclic space* $\mathbb{G}(f) = \operatorname{span}\{U_g f; g \in \mathbb{G}\}$. By the *spectral measure* σ_f of f we mean a Borel measure on $\widehat{\mathbb{G}}$ determined by the equalities

$$\int_{\widehat{\mathbb{G}}} \gamma(g) d\sigma_f(\gamma) = (U_g f, f)$$

for all $g \in \mathbb{G}$.

Theorem 1.1 (spectral theorem). *There exists a sequence $f_1, f_2, \ldots$ in $\mathcal{H}$ such that*

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathbb{G}(f_n) \quad \textit{and} \quad \sigma_{f_1} \gg \sigma_{f_2} \cdots \ . \tag{3}$$

Moreover, for any sequence $f_1', f_2', \ldots$ in $\mathcal{H}$ satisfying (3) we have $\sigma_{f_1} \equiv \sigma_{f_1'}, \sigma_{f_2} \equiv \sigma_{f_2'}, \ldots$ ■

The spectral type of σ_{f_1} (the equivalence class of measures) will be called the *maximal spectral type* of U. U is said to have *Haar spectrum* if $\sigma_{f_1} \equiv \lambda$, where λ is the Haar measure on $\widehat{\mathbb{G}}$. It is said that U has *spectrum of uniform multiplicity* if $\sigma_{f_n} \equiv \sigma_{f_1}$ for $n = 1, 2, \ldots, k$ and $\sigma_{f_n} \equiv 0$ for $n > k$, where $k \in \mathbf{N} \cup \{\infty\}$. We say that an operator U is *mixing* if for any $f, h \in \mathcal{H}$ we have

$$\lim_{g \to \infty} (U_g f, h) = 0.$$

Consider a unitary representation U of the group $\mathbb{G}$ in $L^2(X \times H, \mu \times m)$ given by

$$U_g f(x, h) = f(\boldsymbol{T}_g x, \phi_g(x) h).$$

Let us decompose

$$L^2(X \times H, \mu \times m) = \bigoplus_{\chi \in \widehat{H}} \mathcal{H}_\chi,$$

where

$$\mathcal{H}_\chi = \{f; f(x,h) = \xi(x)\chi(h), \xi \in L^2(X,\mu)\}.$$

Observe that $\mathcal{H}_\chi$ is a closed U–invariant subspace of $L^2(X \times H, \mu \times m)$.

Lemma 1.2. *(see [8]) The representation $U : \mathbb{G} \to \mathcal{U}(\mathcal{H}_\chi)$ is unitarily equivalent to the representation $U_\chi : \mathbb{G} \to \mathcal{U}(L^2(X,\mu))$ given by*

$$((U_\chi)_g\xi)(x) = \chi(\phi_g(x))\xi(\boldsymbol{T}_g x).$$

Proof. We define $V : \mathcal{H}_\chi \to L^2(X,\mu)$ by putting $Vf = \xi$, where $f(x,h) = \xi(x)\chi(h)$. Then V is an isometry from $\mathcal{H}_\chi$ onto $L^2(X,\mu)$ and

$$U_g f(x,h) = f(\boldsymbol{T}_g x, \phi_g(x)h) = \xi(\boldsymbol{T}_g x)\chi(\phi_g(x))\chi(h),$$

so

$$(VU_g f)(x) = \chi(\phi_g(x))\xi(\boldsymbol{T}_g x) = ((U_\chi)_g\xi)(x) = ((U_\chi)_g Vf)(x),$$

and the lemma follows. ■

We say the representation U is *mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$* if U is mixing on the orthocomplement of $\mathcal{H}_1$. We say the representation U has em Haar spectrum of uniform multiplicity on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$ if U has Haar spectrum of uniform multiplicity on the orthocomplement of $\mathcal{H}_1$.

Suppose that $\boldsymbol{T}$ is ergodic and free $\mathbb{G}$–rotation. Let $F : \mathbb{G} \times X \to \mathbb{T}$ be a $\mathbb{T}$–cocycle. Consider a unitary representation of the group $\mathbb{G}$ in $L^2(X,\mu)$ given by

$$(U_g f)(x) = F_g(x) f(\boldsymbol{T}_g x).$$

Lemma 1.3. *(see [8]) The maximal spectral type of U is either discrete or continuous singular or Haar and U has spectrum of uniform multiplicity.* ■

Lemma 1.4. *(see [8]) Suppose that*

$$\lim_{g\to\infty} \int_X F_g(x)d\mu(x) = 0.$$

Then U is mixing. Moreover, if

$$\sum_{g\in\mathbb{G}} |\int_X F_g(x)d\mu(x)|^2 < +\infty,$$

then U has Haar spectrum of uniform multiplicity. ■

Let $\boldsymbol{T}$ be an ergodic and free $\mathbb{Z}^d$–rotation on $\mathbb{T}^d$. Let $\phi : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}$ be a continuous cocycle. For any $q \in \mathbb{Z}$ and $\boldsymbol{m} \in \mathbb{Z}^d$ set

$$s_{\boldsymbol{m},q} = |\int_{\mathbb{T}^d} (\phi(\boldsymbol{z}))^q d\,\boldsymbol{z}\,| = |\int_{\mathbb{T}^d} e^{2\pi i q(h_{\boldsymbol{m}}(\boldsymbol{x}) + \boldsymbol{m}\,W(\phi)\,\boldsymbol{x}^T)} d\,\boldsymbol{x}\,|.$$

By Lemma 1.4, we obtain:

Corollary 1.1. *Suppose that for any $q \in \mathbb{Z} \setminus \{0\}$ we have*

$$\lim_{\boldsymbol{m} \to \infty} s_{\boldsymbol{m},q} = 0.$$

Then the circle extension of $\boldsymbol{T}$ given by

$$\boldsymbol{T}_{\phi} : \mathbb{Z}^d \times \mathbb{T}^d \times \mathbb{T} \to \mathbb{T}^d \times \mathbb{T},\ (\boldsymbol{T}_{\phi})_{\boldsymbol{m}}(\boldsymbol{z}, \omega) = (\boldsymbol{T}_{\boldsymbol{m}}\, \boldsymbol{z}, \phi_{\boldsymbol{m}}(\boldsymbol{z})\omega)$$

is mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$. Moreover, if for any $q \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{\boldsymbol{m} \in \mathbb{Z}^d} s^2_{\boldsymbol{m},q} < \infty,$$

then $\boldsymbol{T}_{\phi}$ has countable Lebesgue spectrum on this orthocomplement. ■

Suppose that $\det W(\phi) \neq 0$. Consider a family of subsets of $\mathbb{Z}^d$ of the form

$$V_l = \{\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\}; |\sum_{j=1}^{d} m_j w_{jl}| = \max_{1 \le k \le d} |\sum_{j=1}^{d} m_j w_{jk}|\},\ \ l = 1, ..., d.$$

Then $\mathbb{Z}^d = \bigcup_{l=1}^{d} V_l \cup \{0\}$. To obtain either mixing or countable Lebesgue spectrum of $\boldsymbol{T}_{\phi}$ it is enough to show that for every $l = 1, ..., d$ and $q \in \mathbb{Z} \setminus \{0\}$ we have either

$$\lim_{\boldsymbol{m} \to \infty, \boldsymbol{m} \in V_l} s_{\boldsymbol{m},q} = 0$$

or

$$\sum_{\boldsymbol{m} \in V_l} s^2_{\boldsymbol{m},q} < \infty$$

respectively. We will need the following simple lemma.

Lemma 1.5. *There exists a constant $C > 0$ such that for any $\boldsymbol{m} \in V_l$ and $k = 1, .., d$ we have*

$$|m_k| \le C |\sum_{j=1}^{d} m_j w_{jl}|.$$

Proof. If $\boldsymbol{m} \in V_l$, then $|c_k| \le |c_l|$, where $c_k = \sum_{j=1}^{d} m_j w_{jk}$ for $k = 1, ..., d$. Put $W = W(\phi)$. By the Cramer's formulas we have

$$|m_k| \le \frac{|\det W_{1k}| + ... + |\det W_{dk}|}{|\det W|} |c_l|.$$

Hence for $C = \sum_{r,s=1}^{d} |\det W_{rs}| / |\det W|$ we obtain

$$|m_k| \le C |\sum_{j=1}^{d} m_j w_{jl}|. \ ■$$

2 Mixing of circle extensions of $\mathbb{Z}^d$–rotations

Let $T : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be an ergodic $\mathbb{Z}^d$–rotation on $\mathbb{R}^d/\mathbb{Z}^d$. For a given $\boldsymbol{m} \in \mathbb{Z}^d$ let the operator $P^T_{\boldsymbol{m}} : L^1(\mathbb{R}^d/\mathbb{Z}^d) \to L^1(\mathbb{R}^d/\mathbb{Z}^d)$ be defined by

$$\frac{f + fT_{\boldsymbol{m}} + fT^2_{\boldsymbol{m}} + ... + fT^{n-1}_{\boldsymbol{m}}}{n} \to P^T_{\boldsymbol{m}} f$$

in $L^1(\mathbb{R}^d/\mathbb{Z}^d)$. By Birkhoff's ergodic theorem, the operator $P^T_{\boldsymbol{m}}$ is well defined and

$$P^T_{\boldsymbol{m}} f \circ T_{\boldsymbol{m}} = P^T_{\boldsymbol{m}} f,\ \int_{\mathbb{T}^d} P^T_{\boldsymbol{m}} f d\boldsymbol{x} = \int_{\mathbb{T}^d} f d\boldsymbol{x},\ P^T_{\boldsymbol{m}}(f \circ T_{\boldsymbol{m'}}) = P^T_{\boldsymbol{m}} f \circ T_{\boldsymbol{m'}}$$

for any $\boldsymbol{m'} \in \mathbb{Z}^d$.

Lemma 2.1. *Let $h : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ be an L^1 cocycle. Then for every $\boldsymbol{m} \in \mathbb{Z}^d$ we have*

$$P^T_{\boldsymbol{m}} h_{\boldsymbol{m}} = \int_{\mathbb{T}^d} h_{\boldsymbol{m}}(\boldsymbol{x}) d\boldsymbol{x}\,.$$

Proof. Since for any $\boldsymbol{m'} \in \mathbb{Z}^d$,

$$h_{\boldsymbol{m}} \circ T_{\boldsymbol{m'}} - h_{\boldsymbol{m}} = h_{\boldsymbol{m'}} \circ T_{\boldsymbol{m}} - h_{\boldsymbol{m'}}$$

we have

$$P^T_{\boldsymbol{m}} h_{\boldsymbol{m}} \circ T_{\boldsymbol{m'}} - P^T_{\boldsymbol{m}} h_{\boldsymbol{m}} = P^T_{\boldsymbol{m}} h_{\boldsymbol{m'}} \circ T_{\boldsymbol{m}} - P^T_{\boldsymbol{m}} h_{\boldsymbol{m'}} = 0.$$

It follows that $P^T_{\boldsymbol{m}} h_{\boldsymbol{m}}$ is T–invariant. By ergodicity of T, $P^T_{\boldsymbol{m}} h_{\boldsymbol{m}}$ is a constant and equal to

$$\int_{\mathbb{T}^d} P^T_{\boldsymbol{m}} h_{\boldsymbol{m}}(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{T}^d} h_{\boldsymbol{m}}(\boldsymbol{x}) d\boldsymbol{x}\,. \blacksquare$$

Definition 2.1. We will say that a function $f : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is *weakly absolutely continuous* (WAC for short) if f is a continuous function and for any $(x_1, ..., x_{j-1}, x_{j+1}, ..., x_d) \in \mathbb{R}^{d-1}$, $j = 1, ..., d$ the function

$$f(x_1, ..., x_{j-1}, \cdot, x_{j+1}, ..., x_d) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$$

is absolutely continuous and for any $j = 1, ..., d$ we have $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^d/\mathbb{Z}^d)$.

Obviously, if f is of class C^1 then f is WAC. We call a cocycle $\phi : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}$ WAC if the cocycle $h : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is WAC.

Lemma 2.2. *Let T be an ergodic $\mathbb{Z}^d$–rotation on $\mathbb{R}^d/\mathbb{Z}^d$. If $h : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is a WAC cocycle, then for every $\boldsymbol{m} \in \mathbb{Z}^d$ and $l = 1, ..., d$ we have*

$$P^T_{\boldsymbol{m}} \frac{\partial}{\partial x_l} h_{\boldsymbol{m}} = 0.$$

In particular, for any $l, j = 1, ..., d$ we have

$$\lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial x_l} h^{(n,j)}_j = 0 \text{ in } L^1(\mathbb{R}^d/\mathbb{Z}^d).$$

Proof. Observe that $\frac{\partial}{\partial x_l}h : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ is an L^1 cocycle. By Lemma 2.1,

$$P^T_{\boldsymbol{m}}\frac{\partial}{\partial x_l}h_{\boldsymbol{m}} = \int_{\mathbb{T}^d}\frac{\partial}{\partial x_l}h_{\boldsymbol{m}}(\boldsymbol{x})d\,\boldsymbol{x} = 0.\blacksquare$$

Theorem 2.3. *Let $\boldsymbol{T}$ be an ergodic $\mathbb{Z}^d$–rotation on $\mathbb{T}^d$ and let $\phi : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}$ be a WAC cocycle. Consider the circle extension of $\boldsymbol{T}$ given by*

$$\boldsymbol{T}_\phi : \mathbb{Z}^d \times \mathbb{T}^d \times \mathbb{T} \to \mathbb{T}^d \times \mathbb{T},\ (\boldsymbol{T}_\phi)_{\boldsymbol{m}}(\boldsymbol{z}, \omega) = (\boldsymbol{T}_{\boldsymbol{m}}\,\boldsymbol{z}, \phi_{\boldsymbol{m}}(\boldsymbol{z})\omega).$$

If $\det W(\phi) \neq 0$, then $\boldsymbol{T}_\phi$ is mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.

Proof. By Corollary 1.1, it is enough to show that for every $l = 1, ..., d$ and $q \in \mathbb{Z}\setminus\{0\}$,

$$\lim_{\boldsymbol{m}\to\infty, \boldsymbol{m}\in V_l} s_{\boldsymbol{m},q} = 0.$$

Applying integration by parts for Stieltjes integrals, for $\boldsymbol{m} \in V_l$ we get

$$\begin{aligned}
&s_{\boldsymbol{m},q}\\
&= |\int_{\mathbb{T}^{d-1}} e^{2\pi iq\sum_{j,k=1;k\neq l}^{d} m_j w_{jk}x_k}(\int_{\mathbb{T}} e^{2\pi iq(h_{\boldsymbol{m}}(\boldsymbol{x})+\sum_{j=1}^{d} m_j w_{jl}x_l)}dx_l)dx_1...\widehat{dx_l}...dx_d|\\
&\le \int_{\mathbb{T}^{d-1}}|\int_{\mathbb{T}} e^{2\pi iq(h_{\boldsymbol{m}}(\boldsymbol{x})+\sum_{j=1}^{d} m_j w_{jl}x_l)}dx_l|dx_1...\widehat{dx_l}...dx_d\\
&= \frac{1}{2\pi|q\sum_{j=1}^d m_j w_{jl}|}\int_{\mathbb{T}^{d-1}}|\int_{\mathbb{T}} e^{2\pi iqh_{\boldsymbol{m}}(\boldsymbol{x})}de^{2\pi iq\sum_{j=1}^d m_j w_{jl}x_l}|dx_1...\widehat{dx_l}...dx_d\\
&= \frac{1}{2\pi|q\sum_{j=1}^d m_j w_{jl}|}\int_{\mathbb{T}^{d-1}}|\int_{\mathbb{T}} e^{2\pi iq\sum_{j=1}^d m_j w_{jl}x_l}de^{2\pi iqh_{\boldsymbol{m}}(\boldsymbol{x})}|dx_1...\widehat{dx_l}...dx_d\\
&= \frac{1}{|\sum_{j=1}^d m_j w_{jl}|}\int_{\mathbb{T}^{d-1}}|\int_{\mathbb{T}} e^{2\pi iq(h_{\boldsymbol{m}}(\boldsymbol{x})+\sum_{j=1}^{d} m_j w_{jl}x_l)}\frac{\partial}{\partial x_l}h_{\boldsymbol{m}}(\boldsymbol{x})dx_l|dx_1...\widehat{dx_l}...dx_d\\
&\le \frac{1}{|\sum_{j=1}^d m_j w_{jl}|}\int_{\mathbb{T}^d}|\frac{\partial}{\partial x_l}h_{\boldsymbol{m}}(\boldsymbol{x})|d\,\boldsymbol{x}\\
&\le \sum_{k=1}^d\frac{|m_k|}{|\sum_{j=1}^d m_j w_{jl}|}\int_{\mathbb{T}^d}|\frac{\frac{\partial}{\partial x_l}h_k^{(m_k,k)}(\boldsymbol{x})}{m_k}|d\,\boldsymbol{x}.
\end{aligned}$$

For $n \in \mathbb{Z}\setminus\{0\}$ set

$$b_n = \max_{1\le k\le d}\int_{\mathbb{T}^d}|\frac{\frac{\partial}{\partial x_l}h_k^{(n,k)}(\boldsymbol{x})}{n}|d\,\boldsymbol{x}.$$

Then $b_{-n} = b_n$. By Lemma 2.2, $\lim\limits_{n\to\infty} b_n = 0$.

If the sequence $\{nb_n\}_{n\in\mathbb{N}}$ is bounded by $M > 0$, then

$$|\int_{\mathbb{T}^d} e^{2\pi iq(h_{\boldsymbol{m}}(\boldsymbol{x})+\boldsymbol{m}\,W(\phi)\,\boldsymbol{x}^T)}d\,\boldsymbol{x}\,| \le \sum_{k=1}^d\frac{|m_k b_{m_k}|}{|\sum_{j=1}^d m_j w_{jl}|} \le d^2MC\frac{1}{\sum_{l=1}^d|m_k|},$$

by Lemma 1.5. Since $\lim_{\boldsymbol{m}\to\infty} 1/\sum_{l=1}^{d} |m_k| = 0$, we obtain

$$\lim_{\boldsymbol{m}\to\infty, \boldsymbol{m}\in V_l} \Big| \int_{\mathbb{T}^d} e^{2\pi i q(h_{\boldsymbol{m}}(\boldsymbol{x})+\boldsymbol{m}\, W(\phi)\,\boldsymbol{x}^T)} d\,\boldsymbol{x} \Big| = 0.$$

Suppose now that the sequence $\{nb_n\}_{n\in\mathbb{Z}}$ is unbounded. Fix $\varepsilon > 0$. We have to show that there exists a constant $R > 0$ such that if $\boldsymbol{m} = (m_1, ..., m_d) \in \boldsymbol{m} \in V_l$ and $\max(|m_1|, ..., |m_d|) > R$, then

$$\Big| \int_{\mathbb{T}^d} e^{2\pi i q(h_{\boldsymbol{m}}(\boldsymbol{x})+\boldsymbol{m}\, W(\phi)\,\boldsymbol{x}^T)} d\,\boldsymbol{x} \Big| < \varepsilon.$$

Let n_0 be a natural number such that for $|n| \geq n_0$ we have $b_n < \frac{\varepsilon}{2dC}$. Set

$$R = \min\{r \in \mathbb{N}; r \geq n_0, \max_{|n|\leq r} |nb_n| \leq rb_r\}.$$

Then for $|n| > R$ we have $b_n < \frac{\varepsilon}{2dC}$. If $\boldsymbol{m} \in V_l$ and $\max(|m_1|, ..., |m_d|) > R$, then the set $D = \{k \in \{1, ..., d\}; |m_k| > R\}$ is not empty. Choose $k_0 \in D$. Applying Lemma 1.5 we obtain

$$\begin{aligned} s_{\boldsymbol{m},q} &\leq \sum_{k=1}^{d} \frac{|m_k b_{m_k}|}{|\sum_{j=1}^{d} m_j w_{jl}|} \leq \sum_{k\in D} C b_{m_k} + \sum_{k\notin D} C \frac{|m_k b_{m_k}|}{m_{k_0}} \\ &\leq \varepsilon/2 + \sum_{k\notin D} C \frac{R b_R}{R} \leq \varepsilon/2 + \sum_{k\notin D} C b_R < \varepsilon, \end{aligned}$$

which completes the proof. ∎

Corollary 2.1. *If ϕ is of class C^1 and $\det W(\phi) \neq 0$, then $\boldsymbol{T}_\phi$ is mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.*

3 On functions of bounded variation on I^2

Let $I = [0, 1]$ and $I^2 = [0, 1] \times [0, 1]$. In this section we will study some properties of functions of bounded variation on I^2. It will be useful to obtain countable Lebesgue spectrum of $\boldsymbol{T}_\phi$ in the case when $d = 2$.

For a closed rectangle $Q = [a_1, a_2] \times [b_1, b_2] \subset I^2$ the linear functional $\Delta_Q^* : \mathbb{C}^{I^2} \to \mathbb{C}$ is defined by

$$\Delta_Q^* f = f(a_2, b_2) - f(a_1, b_2) - f(a_2, b_1) + f(a_1, b_1).$$

By a *partition* P of I^2, we mean a partition into rectangles $[\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)}] \times [\eta_{i_2}^{(2)}, \eta_{i_2+1}^{(2)}]$ given by sequences

$$\{(\eta_0^{(j)}, \eta_1^{(j)}, ..., \eta_{m_j}^{(j)}); 0 = \eta_0^{(j)} \leq ... \leq \eta_{m_j}^{(j)} = 1\ ,\ j = 1, 2\}.$$

Given such a partition, for $i_1 = 0, ..., m_1 - 1$ and $i_2 = 0, ..., m_2 - 1$, the linear functional $\Delta^{i_1 i_2} : \mathbb{C}^{I^2} \to \mathbb{C}$ is defined by

$$\begin{aligned} \Delta^{i_1 i_2} f &= \Delta^*_{[\eta_{i_1}^{(1)}, \eta_{i_1+1}^{(1)}]\times[\eta_{i_2}^{(2)}, \eta_{i_2+1}^{(2)}]} f \\ &= f(\eta_{i_1+1}^{(1)}, \eta_{i_2+1}^{(2)}) - f(\eta_{i_1+1}^{(1)}, \eta_{i_2}^{(2)}) - f(\eta_{i_1}^{(1)}, \eta_{i_2+1}^{(2)}) + f(\eta_{i_1}^{(1)}, \eta_{i_2}^{(2)}). \end{aligned}$$

Definition 3.1. For a function $f : I^2 \to \mathbb{C}$, by the *variation* of f we mean

$$\mathrm{Var}^{(2)} f = \sup_{P \in \mathcal{P}} \sum_{i_1=0}^{m_1-1} \sum_{i_2=0}^{m_2-1} |\Delta^{i_1 i_2} f|,$$

where $\mathcal{P}$ is the family of all partitions P of I^2. If $\mathrm{Var}^{(2)} f$ is finite, then f is said to be *of bounded variation on I^2 in the sense of Vitali.*

Definition 3.2. A function f is said to be *of bounded variation on I^2 in the sense of Hardy and Krause* if f is of bounded variation in the sense of Vitali and both of the functions $f(0,\cdot), f(\cdot,0) : I \to \mathbb{C}$ are of bounded variation in the ordinary sense.

In what follows functions of bounded variation are those of bounded variation in the sense of Hardy and Krause. We will denote by BV the space of all functions of bounded variation on I^2. We will consider the norm on BV given by

$$\|f\|_{BV} = \sup_{\boldsymbol{x} \in I^2} |f(\boldsymbol{x})| + \mathrm{Var}_I f(\cdot, 0) + \mathrm{Var}_I f(0, \cdot) + \mathrm{Var}^{(2)} f.$$

Recall that if a function is of bounded variation, then it is integrable in the sense of Riemann (see [6] §448).

For $m \in \mathbb{Z}$ set $|m|_1 = \max(|m|, 1)$.

Lemma 3.1. *Let $f : I^2 \to \mathbb{C}$ be a function on bounded variation. If $g : I \to \mathbb{C}$ is a function given by $g(t) = f(\{pt + c\}, \{qt + d\})$, where $p, q \in \mathbb{Z}$, $c, d \in \mathbb{R}$, then*

$$\mathrm{Var}_I g \leq |p|_1 |q|_1 \|f\|_{BV}. \ \blacksquare$$

Let $f, g : I^2 \to \mathbb{C}$ be bounded functions. We will denote by $\int_{I^2} f dg$ the Riemann-Stieltjes integral of function f with respect to g (see [6] §381). Recall that if both f and g are of bounded variation and if at least one of the functions is continuous then $\int_{I^2} f dg$ exists (see [6] §448) and

$$|\int_{I^2} f dg| \leq \sup_{x \in I^2} |f(x)| \mathrm{Var}^{(2)} g. \tag{4}$$

Theorem 3.2 (integration by parts). *(See [6] §448.) Let $f, g : I^2 \to \mathbb{C}$ be functions of bounded variation and let at least one of them be continuous. Then*

$$\begin{aligned} \int_{I^2} f dg &= \int_{I^2} g df - \int_I g(\cdot, 1) df(\cdot, 1) + \int_I g(\cdot, 0) df(\cdot, 0) \\ &\quad - \int_I g(1, \cdot) df(1, \cdot) + \int_I g(0, \cdot) df(0, \cdot) + \Delta^*_{I^2} g f. \blacksquare \end{aligned}$$

We say $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ is of bounded variation if $f|_{I^2}$ is of bounded variation. For any $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$, $a, b \in \mathbb{R}$ set $f_{a,b}(x_1, x_2) = (x_1 + a, x_2 + b)$. Then $\mathrm{Var}^{(2)} f_{a,b} = \mathrm{Var}^{(2)} f$. By the previous theorem, we obtain:

Corollary 3.1. *Let $f, g : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ be functions of bounded variation and let at least one of them be continuous. Then*

$$\int_{I^2} f dg = \int_{I^2} g df.$$

Lemma 3.3. *If $f, g \in BV$, then $fg \in BV$ and we have*

$$\|fg\|_{BV} \leq 2\|f\|_{BV}\|g\|_{BV}. \blacksquare$$

Recall that if $f : I \to \mathbb{C}$ is of bounded variation and there exists a real number $a > 0$ such that $0 < a \leq |f(x)|$ for any $x \in I$, then the function $1/f$ is of bounded variation and

$$\text{Var}_I(\frac{1}{f}) \leq \frac{\text{Var}_I f}{a^2}. \tag{5}$$

Lemma 3.4. *Let $f \in BV$ and assume that there exists a real number a such that for every $x \in I^2$ we have $0 < a \leq |f(\boldsymbol{x})|$. Then $1/f \in BV$ and*

$$\text{Var}^{(2)} \frac{1}{f} \leq \frac{\|f\|_{BV}}{a^2} + \frac{2\|f\|_{BV}^2}{a^3}. \blacksquare$$

Definition 3.3. We say that a function $f : I^d \to \mathbb{C}$ is *differentiable in the sense of Vitali* at $(x_1, x_2) \in I^2$ if

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\Delta^*_{[x_1,x_1+h_1]\times[x_2,x_2+h_2]} f}{h_1 h_2},$$

exists. This limit is called the *derivative* of f and is denoted by $Df(x_1, x_2)$.

Remark. If $f \in C^2(I^2)$, then $Df(\boldsymbol{x}) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{x})$ (see [12] ch.7 §1). If a function f is of bounded variation in the sense of Vitali, then f is differentiable in the sense of Vitali almost everywhere (see [12] ch.7 §2).

Definition 3.4. A function f is said to be *differentiable in the sense of Hardy and Krause* at $\boldsymbol{x} \in I^2$ if f is differentiable in the sense of Vitali and the partial derivatives of f at $\boldsymbol{x}$ exist.

In what follows, by differentiable functions we mean those which are differentiable in the sense of Hardy and Krause.

Lemma 3.5. *Let $f : I^2 \to \mathbb{C}$ be a differentiable function. Then the function* $\exp f : I^2 \to \mathbb{C}$ *is differentiable and we have*

$$D \exp f(\boldsymbol{x}) = \exp f(\boldsymbol{x})(Df(\boldsymbol{x}) + \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \frac{\partial}{\partial x_2} f(\boldsymbol{x})). \blacksquare$$

The number $|P| = (b_1 - a_1)(b_2 - a_2)$ is called the *substance* of the rectangle $P = [a_1, b_1] \times [a_2, b_2]$.

Definition 3.5. A function $f : I^2 \to \mathbb{C}$ is said to be *absolutely continuous in the sense of Vitali* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every system of rectangles $Q_1, ..., Q_n$ such that Int Q_i are pairwise disjoint,

$$|Q_1| + ... + |Q_n| < \delta \Longrightarrow |\, \Delta^*_{Q_1} f \,| + ... + |\, \Delta^*_{Q_n} f \,| < \varepsilon.$$

Remark. If a function is absolutely continuous in the sense of Vitali, then it is of bounded variation in the sense of Vitali (see [12] ch.7 §3).

Definition 3.6. A function f is said to be *is absolutely continuous on* I^2 *in the sense of Hardy and Krause if* f *is absolutely continuous in the sense of Vitali and both of the functions* $f(0,\cdot), f(\cdot,0) : I \to \mathbb{C}$ *are absolutely continuous in the ordinary sense.*

In what follows absolutely continuous functions are those absolutely continuous in the sense of Hardy and Krause. We will denote by AC the space of function which are absolutely continuous on I^2. A function $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ is absolutely continuous if $f|_{I^2}$ is absolutely continuous.

Recall that (see [12] ch.7 §3) if a function f is of bounded variation and g is absolutely continuous, then

$$\int_{I^2} f dg = \int_{I^2} f Dg d\boldsymbol{x} \,. \tag{6}$$

Lemma 3.6. *Let* $f : I^2 \to \mathbb{R}$ *be an absolutely continuous function such that* $f(x_1,1)-f(x_1,0), f(1,x_2)-f(0,x_2) \in \mathbb{Z}$ *for any* $(x_1,x_2) \in I^2$ *and* $Df, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \in BV$. *Suppose that there exists a real number* $a > 0$ *such that*

$$|Df(\boldsymbol{x}) - 2\pi i \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \frac{\partial}{\partial x_2} f(\boldsymbol{x})| \geq a > 0$$

for any $\boldsymbol{x} \in I^2$. *Then*

$$\begin{aligned} |\int_{I^2} \exp 2\pi i f(\boldsymbol{x}) d\boldsymbol{x}\,| \quad \leq \quad & \frac{\|Df\|_{BV} + 2\|\frac{\partial}{\partial x_1} f\|_{BV} \|\frac{\partial}{\partial x_2} f\|_{BV}}{a^2} \\ & + \frac{\|Df\|^2_{BV} + 16\pi \|\frac{\partial}{\partial x_1} f\|^2_{BV} \|\frac{\partial}{\partial x_2} f\|^2_{BV}}{a^3}. \end{aligned}$$

Proof. An application in succession (6), Lemma 3.5, integration by parts, (4) and Lemma 3.4 gives that

$$\begin{aligned} |\int_{I^2} \exp 2\pi i f(\boldsymbol{x}) d\boldsymbol{x}\,| \quad &= \quad \frac{1}{2\pi} |\int_{I^2} \frac{1}{Df - 2\pi i \frac{\partial}{\partial x_1} f \frac{\partial}{\partial x_2} f} de^{2\pi i f}| \\ &= \quad \frac{1}{2\pi} |\int_{I^2} e^{2\pi i f} d \frac{1}{Df - 2\pi i \frac{\partial}{\partial x_1} f \frac{\partial}{\partial x_2} f}| \\ &\leq \quad \frac{1}{2\pi} \mathrm{Var}^{(2)} \frac{1}{Df - 2\pi i \frac{\partial}{\partial x_1} f \frac{\partial}{\partial x_2} f} \\ &\leq \quad \frac{\|Df - 2\pi i \frac{\partial}{\partial x_1} f \frac{\partial}{\partial x_2} f\|_{BV}}{2\pi a^2} + \frac{\|Df - 2\pi i \frac{\partial}{\partial x_1} f \frac{\partial}{\partial x_2} f\|^2_{BV}}{\pi a^3} \\ &\leq \quad \frac{\|Df\|_{BV} + 2\|\frac{\partial}{\partial x_1} f\|_{BV} \|\frac{\partial}{\partial x_2} f\|_{BV}}{a^2} \\ & \quad + \frac{\|Df\|^2_{BV} + 16\pi \|\frac{\partial}{\partial x_1} f\|^2_{BV} \|\frac{\partial}{\partial x_2} f\|^2_{BV}}{a^3}, \end{aligned}$$

and the proof is complete. ■

4 Koksma inequalities and Diophantine approximation on the torus

Definition 4.1. Let $\boldsymbol{x}_1, ..., \boldsymbol{x}_N$ be a sequence in $\mathbb{R}^d$. By the *discrepancy* of

$\boldsymbol{x}_1, ..., \boldsymbol{x}_N$ we mean

$$D_N^*(\boldsymbol{x}_1, ..., \boldsymbol{x}_N) = \sup_{J \in \mathcal{J}} |\frac{1}{N} \sum_{n=1}^{N} \chi_J(\{\boldsymbol{x}_n\}) - \lambda(J)|,$$

where $\mathcal{J}$ is the family of subcubes of I^d of the form $\prod_{i=1}^d [0, \beta_i)$, where $0 \leq \beta_i < 1$ for $j = 1, ..., d$ and $\{\boldsymbol{x}\} = (\{x_1\}, ..., \{x_d\})$ for $\boldsymbol{x} = (x_1, ..., x_d)$.

Remark. If $\gamma_1, ..., \gamma_d, 1$ are independent over $\mathbb{Q}$, then $\lim_{N \to \infty} D_N^*(\{n\boldsymbol{\gamma}\}_{n=1}^N) = 0$, where $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_d)$.

Set $\|x\| = \inf_{p \in \mathbb{Z}} |x + p| = \min(\{x\}, 1 - \{x\})$ for any $x \in \mathbb{R}$. For any $\boldsymbol{h} = (h_1, ..., h_d) \in \mathbb{Z}^d$ and $\boldsymbol{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ denote

$$< \boldsymbol{h}, \boldsymbol{x} >= \sum_{j=1}^{d} h_j x_j \text{ and } |\boldsymbol{h}| = \prod_{j=1}^{d} \max(|h_j|, 1).$$

Definition 4.2. Let $\gamma_1, ..., \gamma_d, 1$ be real numbers independent over $\mathbb{Q}$. The multinumber $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_d)$ is called of *type* $\eta \geq 1$ if there exists $C > 0$ such that for any $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{0\}$

$$\| < \boldsymbol{h}, \boldsymbol{\gamma} > \| \geq \frac{C}{|\boldsymbol{h}|^\eta}$$

We say $\boldsymbol{\gamma}$ is of *finite type* if there exists $\eta \geq 1$ such that $\boldsymbol{\gamma}$ is of type η.

It follows from the definition that for any $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\boldsymbol{h} \in \mathbb{Z}^d$ and $m \in \mathbb{Z} \setminus \{0\}$ the multinumber $\boldsymbol{\gamma}$ is of type η iff $-\boldsymbol{\gamma}$ is of type η iff $\boldsymbol{\gamma} + \boldsymbol{h}$ is of type η iff $m\boldsymbol{\gamma}$ is of type η.

Lemma 4.1. *(See [10]) If $\gamma \in \mathbb{R}$, $(\gamma_1, \gamma_2) \in \mathbb{R}^2$ are of type η, then there exists $L > 0$ such that*

$$D_N^*(\{n\gamma\}_{n=1}^N) \leq \frac{L}{N^{1/\eta}},$$

$$D_N^*(\{(n\gamma_1, n\gamma_2)\}_{n=1}^N) \leq \frac{L \log N}{N^{1/(2\eta-1)}}. \blacksquare$$

If $\gamma_1, \gamma_2, 1$ are linearly dependent over $\mathbb{Q}$ and $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \mathbb{Q}$ then there exist $t_1, t_2, t_3 \in \mathbb{Z}$, $t_1, t_2 \neq 0$ such that $t_1\gamma_1 + t_2\gamma_2 = t_3$. Take $s_1, s_2 \in \mathbb{Z}$ such that $t_2 s_1 - t_1 s_2 = \gcd(t_1, t_2)$.

Definition 4.3. Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ be a pair such that at least one of the numbers γ_1, γ_2 is irrational. Then the pair $\boldsymbol{\gamma}$ is called of *type* η

(i) for $\gamma_1, \gamma_2, 1$ rationally independent if $\boldsymbol{\gamma}$ is of type η in the ordinary sense (Definition 4.2),

(ii) for $\gamma_1, \gamma_2, 1$ rationally dependent and $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \mathbb{Q}$ if $s_1\gamma_1 + s_2\gamma_2$ is of type η in the ordinary sense,

(iii) for $\gamma_2 \in \mathbb{Q}$ if γ_1 is of type η in the ordinary sense.

Note that the second part of this definition is independent of the choice of t_1, t_2, s_1, s_2.

Theorem 4.2 (Denjoy-Koksma inequality). *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be of bounded variation and let $\{x_n\}_{n=1}^N$ be a sequence of real numbers. Then*

$$|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_I f(x)dx| \leq D_N^*(\{x_n\}_{n=1}^N)\mathrm{Var}_I f.$$

Theorem 4.3 (Koksma-Hlawka inequality). *Let $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ be of bounded variation and let $\{(x_n^{(1)}, x_n^{(2)})\}_{n=1}^N$ be a sequence in $\mathbb{R}^2$. Then*

$$\begin{aligned} |\frac{1}{N}\sum_{n=1}^{N} f(x_n^{(1)}, x_n^{(2)}) - \int_{\mathbb{T}^2} f(\boldsymbol{x})d\boldsymbol{x}| \quad &\leq \quad D_N^*(\{x_n^{(1)}\}_{n=1}^N)\mathrm{Var}_I f(\cdot, 1) \\ &\quad + D_N^*(\{x_n^{(2)}\}_{n=1}^N)\mathrm{Var}_I f(1, \cdot) \\ &\quad + D_N^*(\{(x_n^{(1)}, x_n^{(2)})\}_{n=1}^N)\mathrm{Var}^{(2)} f. \end{aligned}$$

The proofs of the above theorems can be found in [10].

Theorem 4.4. *Let $\boldsymbol{\gamma} \in \mathbb{R}^2$ be of type η. Then there exists a linear operator $P_\gamma : L^1(\mathbb{R}^2/\mathbb{Z}^2) \to L^1(\mathbb{R}^2/\mathbb{Z}^2)$ and a constant $L > 0$ such that for any function $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ of bounded variation and for any natural $N \geq 2$ we have*

$$|\frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x} + n\boldsymbol{\gamma}) - P_\gamma f(\boldsymbol{x})| \leq \frac{L\log N}{N^{1/(2\eta-1)}}\|f\|_{BV}.$$

Moreover, $P_\gamma f(\boldsymbol{x} + \boldsymbol{\gamma}) = P_\gamma f(\boldsymbol{x})$ and $\int_{I^2} P_\gamma f(\boldsymbol{x})d\boldsymbol{x} = \int_{I^2} f(\boldsymbol{x})d\boldsymbol{x}$.

Proof. We will use the symbol S_N to denote

$$|\frac{1}{N}\sum_{n=0}^{N-1} f(\boldsymbol{x} + n\boldsymbol{\gamma}) - P_\gamma f(\boldsymbol{x})|.$$

CASE 1. Suppose that $\gamma_1, \gamma_2, 1$ are linearly independent over $\mathbb{Q}$. Put $P_\gamma f(\boldsymbol{x}) = \int_{\mathbb{T}^2} f d\lambda$. By the Koksma-Hlawka inequality and Lemma 4.1,

$$\begin{aligned} S_N \quad &\leq \quad \frac{L_1}{N^{1/\eta}}(\mathrm{Var}_I f(\cdot, 1) + \mathrm{Var}_I f(1, \cdot)) + \frac{L_1 \log N}{N^{1/(2\eta-1)}}\mathrm{Var}^{(2)} f \\ &\leq \quad \frac{L\log N}{N^{1/(2\eta-1)}}\|f\|_{BV}, \end{aligned}$$

where L_1 is the constant from Lemma 4.1 and $L = 3L_1$.

CASE 2. Suppose that $\gamma_2 = \frac{p}{q}$, where $p \in \mathbb{Z}, q \in \mathbb{N}$. By the Denjoy-Koksma inequality and Lemma 4.1, for any natural M and $(x_1, x_2) \in \mathbb{T}^2$ we have

$$|\frac{1}{M}\sum_{n=0}^{M-1} f(x_1 + nq\gamma_1, x_2 + np) - \int_{\mathbb{T}} f(x, x_2)dx| \leq \frac{L_1}{M^{1/\eta}}\mathrm{Var}_I f(\cdot, x_2).$$

Replace x_2 by $x_2 + jp/q$ and f by $f_{j\gamma_1,0}$ for $j = 0, ..., q-1$. Sum up these q inequalities. Dividing the resulting inequality by q we get

$$\begin{aligned} &|\frac{1}{qM}\sum_{n=0}^{qM-1} f(x_1 + n\gamma_1, x_2 + n\frac{p}{q}) - \frac{1}{q}\sum_{j=1}^{q}\int_{\mathbb{T}} f(x, x_2 + j\frac{p}{q})dx| \\ &\leq \frac{L_1}{qM^{1/\eta}}\sum_{j=1}^{q}\mathrm{Var}_I f(\cdot, x_2 + j\frac{p}{q}). \end{aligned}$$

For any natural N choose a natural number M such that $qM \le N < (M+1)q$. Put $P_\gamma f(x_1,x_2) = \frac{1}{q}\sum_{j=1}^{q}\int_{\mathbb{T}} f(x, x_2 + jp/q)dx$. Then

$$\begin{aligned} S_N = & \ |\frac{1}{N}\sum_{n=0}^{N-1} f(x_1+n\gamma_1, x_2+n\frac{p}{q}) - \frac{1}{q}\sum_{j=1}^{q}\int_{\mathbb{T}} f(x, x_2+j\frac{p}{q})dx| \\ \le & \ |\frac{1}{qM}\sum_{n=0}^{qM-1} f(x_1+n\gamma_1, x_2+n\frac{p}{q}) - \frac{1}{q}\sum_{j=1}^{q}\int_{\mathbb{T}} f(x, x_2+j\frac{p}{q})dx| \\ & +|\frac{qM-N}{NqM}\sum_{n=0}^{qM-1} f(x_1+n\gamma_1, x_2+n\frac{p}{q}) + \frac{1}{N}\sum_{n=qM}^{N-1} f(x_1+n\gamma_1, x_2+n\frac{p}{q})| \\ \le & \ \frac{L_1}{qM^{1/\eta}}\sum_{j=1}^{q}\mathrm{Var}_I f(\cdot, x_2+j\frac{p}{q}) + \frac{2(N-qM)}{N}\sup_{\boldsymbol{x}\in\mathbb{T}^2}|f(\boldsymbol{x})|. \end{aligned}$$

Since $N - qM < q$ and $N < 2qM$,

$$S_N \le \frac{2q}{N}\sup_{\boldsymbol{x}\in\mathbb{T}^2}|f(\boldsymbol{x})| + \frac{2L_1}{N^{1/\eta}}\sum_{j=1}^{q}\mathrm{Var}_I f(\cdot, x_2+j\frac{p}{q}). \tag{7}$$

By Lemma 3.1, we obtain

$$S_N \le \frac{L\log N}{N^{1/(2\eta-1)}}\|f\|_{BV},$$

where $L = 2q(L_1+1)$.

CASE 3. Suppose that $\gamma_1, \gamma_2, 1$ are linearly dependent over $\mathbb{Q}$ and $\gamma_1, \gamma_2 \in \mathbb{R}\setminus\mathbb{Q}$. Then there exist $t_1, t_2, t_3 \in \mathbb{Z}$, $t_1, t_2 \ne 0$ such that $t_1\gamma_1 + t_2\gamma_2 = t_3$. Take $s_1, s_2 \in \mathbb{Z}$ such that $t_2 s_1 - t_1 s_2 = \gcd(t_1, t_2)$. Set $t = \gcd(t_1, t_2)$ and

$$B = \begin{bmatrix} s_1 & s_2 \\ t_1/t & t_2/t \end{bmatrix}.$$

Since $B \in M_2(\mathbb{Z})$ and $\det B = 1$, $B : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a group automorphism. Consider the function $g : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}$ given by $g = fB^{-1}$. Then the function $g(\cdot, x_2) : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is of bounded variation for any $x_2 \in \mathbb{R}$. Replace γ_1 by $s_1\gamma_1 + s_2\gamma_2$, f by g, q by t, p by t_3 in (7). Applying Lemma 3.1 we obtain

$$|\frac{1}{N}\sum_{n=0}^{N-1} g(y_1 + n(s_1\gamma_1+s_2\gamma_2), y_2 + n\frac{t_3}{t}) - \frac{1}{t}\sum_{j=1}^{t}\int_{\mathbb{T}} g(x, y_2+j\frac{t_3}{t})dx| \le$$

$$\frac{2t}{N}\sup_{\boldsymbol{x}\in\mathbb{T}^2}|g(\boldsymbol{x})| + \frac{2L_1}{N^{1/\eta}}\sum_{j=1}^{t}\mathrm{Var}_I g(\cdot, y_2+j\frac{t_3}{t}) \le \frac{L\log N}{N^{1/(2\eta-1)}}\|f\|_{BV},$$

for any $(y_1, y_2) \in \mathbb{R}^2$, where $L = 2t(1 + L_1|t_1||t_2|)$. Put

$$P_\gamma f(x_1, x_2) = \frac{1}{t}\sum_{j=1}^{t}\int_{\mathbb{T}} fB^{-1}(x, \frac{t_1x_1 + t_2x_2 + jt_3}{t})dx.$$

With notation $y_1 = s_1x_1 + s_2x_2$ and $y_2 = (t_1x_1 + t_2x_2)/t$ we have

$$\begin{aligned}
S_N &= |\frac{1}{N}\sum_{n=0}^{N-1} f(x_1+n\gamma_1, x_2+n\gamma_2) - \frac{1}{t}\sum_{j=1}^{t}\int_{\mathbb{T}} fB^{-1}(x, \frac{t_1x_1+t_2x_2+jt_3}{t})dx| \\
&= |\frac{1}{N}\sum_{n=0}^{N-1} gB(x_1+n\gamma_1, x_2+n\gamma_2) - \frac{1}{t}\sum_{j=1}^{t}\int_{\mathbb{T}} g(x, \frac{t_1x_1+t_2x_2+jt_3}{t})dx| \\
&= |\frac{1}{N}\sum_{n=0}^{N-1} g(y_1+n(s_1\gamma_1+s_2\gamma_2), y_2+n\frac{t_3}{t}) - \frac{1}{t}\sum_{j=1}^{t}\int_{\mathbb{T}} g(x, y_2+j\frac{t_3}{t})dx| \\
&\leq \frac{L\log N}{N^{1/(2\eta-1)}}\|f\|_{BV}. \blacksquare
\end{aligned}$$

5 Spectral properties of extensions of $\mathbf{Z}^2$–rotations

Let $\boldsymbol{T} : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ be an ergodic and free $\mathbb{Z}^2$–rotation given by

$$\boldsymbol{T}_{(m_1,m_2)}(z_1, z_2) = (e^{2\pi i(\alpha_{11}m_1+\alpha_{12}m_2)}z_1, e^{2\pi i(\alpha_{21}m_1+\alpha_{22}m_2)}z_2).$$

Definition 5.1. We say the rotation $\boldsymbol{T}$ is of *type* η if both of the pair $(\alpha_{11}, \alpha_{21})$, $(\alpha_{12}, \alpha_{22})$ are of type η. The rotation $\boldsymbol{T}$ said to be of *finite type* if there exists $\eta \geq 1$ such that $\boldsymbol{T}$ is of type η.

Lemma 5.1. *Suppose that $\boldsymbol{T}$ is of type η. There exists a constant $L > 0$ such that if $h : \mathbb{Z}^d \times \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d$ is an absolutely continuous cocycle and the cocycles $\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}$ are of bounded variation, then*

$$|\frac{1}{N}\frac{\partial}{\partial x_k}h_j^{(N,j)}| \leq \frac{L\log|N|}{|N|^{1/(2\eta-1)}}\|\frac{\partial}{\partial x_k}h_j\|_{BV} \tag{8}$$

for any $|N| \geq 2$ and $j, k = 1, 2$.

Proof. It suffices to show that the inequality (8) is true for any natural N. By Theorem 4.4,

$$|\frac{1}{N}\frac{\partial}{\partial x_k}h_j^{(N,j)} - P_{(\alpha_{1j},\alpha_{2j})}(\frac{\partial}{\partial x_k}h_j)| \leq \frac{L\log N}{N^{1/(2\eta-1)}}\|\frac{\partial}{\partial x_k}h_j\|_{BV}.$$

Observe that $P^{T}_{\boldsymbol{m}} = P_{\boldsymbol{\alpha}\boldsymbol{m}}$. Application Lemma 2.2 gives

$$P_{(\alpha_{1j},\alpha_{2j})}(\frac{\partial}{\partial x_k}h_j) = 0. \blacksquare$$

Theorem 5.2. *Suppose that $\boldsymbol{T}$ is an ergodic and free $\mathbb{Z}^2$–rotation on $\mathbb{T}^2$ which is of finite type. Let $\phi : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}$ be an absolutely continuous cocycle such that the cocycles $Dh, \frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}$ are of bounded variation. If $\det W(\phi) \neq 0$, then $\boldsymbol{T}_{\phi}$ has countable Lebesgue spectrum on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.*

In view of Corollary 1.1 it is enough to prove that for any $q \in \mathbb{Z} \setminus \{0\}$ and $l = 1, 2$,

$$\sum_{\boldsymbol{m}\in V_l} s^2_{\boldsymbol{m},q} < \infty,$$

where

$$\begin{aligned} s_{\boldsymbol{m},q} &= \left|\int_{\mathbb{T}^2} \phi_{\boldsymbol{m}}(\boldsymbol{z})^q d\boldsymbol{z}\right| \\ &= \left|\int_{\mathbb{T}^2} e^{2\pi i q(h_{\boldsymbol{m}}(x_1,x_2)+(m_1w_{11}+m_2w_{21})x_1+(m_1w_{12}+m_2w_{22})x_2)} dx_1dx_2\right|. \end{aligned}$$

We will prove the above condition only for $l=1$. The proof in the case $l=2$ is similar.

Let η be a type of $\boldsymbol{T}$. Take a real number a such that $\max(1-1/(2\eta-1), 3/4) < a < 1$. Then there exists $K>0$ such that

$$\frac{\log N}{N^{1/(2\eta-1)}} \leq K\frac{N^a}{N} \tag{9}$$

for any natural $N \geq 2$. Set

$$M = 1 + KLC^a\left(\left\|\frac{\partial}{\partial x_1}h_1\right\|_{BV} + \left\|\frac{\partial}{\partial x_1}h_2\right\|_{BV} + \left\|\frac{\partial}{\partial x_2}h_1\right\|_{BV} + \left\|\frac{\partial}{\partial x_2}h_2\right\|_{BV}\right),$$

where C and L are the constants from Lemma 1.5 and Lemma 5.1 respectively.

Lemma 5.3. *There exists a constant $R>0$ such that if $\boldsymbol{m} \in V_1$ and*

$$(2M)^{1/(1-a)} \leq |m_1w_{11} + m_2w_{21}|,$$

then

$$s_{\boldsymbol{m},q} \leq \frac{R}{|m_1w_{11} + m_2w_{21}|}.$$

Proof. Applying Lemma 5.1, Lemma 1.5 and inequality (9) for any $\boldsymbol{x} \in \mathbb{R}^2/\mathbb{Z}^2$ we get

$$\begin{aligned} \frac{|\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x})|}{|m_1w_{11}+m_2w_{21}|} &\leq \frac{|\frac{\partial}{\partial x_1}h_1^{(m_1,1)}(\boldsymbol{T}_{(0,m_2)}\boldsymbol{x})| + |\frac{\partial}{\partial x_1}h_2^{(m_2,2)}(\boldsymbol{x})|}{|m_1w_{11}+m_2w_{21}|} \\ &\leq KL\frac{|m_1|^a\|\frac{\partial}{\partial x_1}h_1\|_{BV} + |m_2|^a\|\frac{\partial}{\partial x_1}h_2\|_{BV}}{|m_1w_{11}+m_2w_{21}|} \\ &\leq \frac{M}{|m_1w_{11}+m_2w_{21}|^{1-a}}. \end{aligned}$$

It follows that for $(2M)^{1/(1-a)} \leq |m_1w_{11}+m_2w_{21}|$ we have

$$\left|\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x})\right| \leq \frac{1}{2}|m_1w_{11}+m_2w_{21}|$$

and

$$\left|\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x}) + m_1w_{11}+m_2w_{21}\right| \geq \frac{1}{2}|m_1w_{11}+m_2w_{21}|. \tag{10}$$

Applying in succession integration by parts, (5) with (10), Lemma 3.1 and

Lemma 1.5 we obtain

$$\begin{aligned}
&s_{\boldsymbol{m},q}\\
&= |\int_{\mathbb{T}} e^{2\pi i q(m_1w_{12}+m_2w_{22})x_2}(\int_{\mathbb{T}} e^{2\pi i q(h_{\boldsymbol{m}}(x_1,x_2)+(m_1w_{11}+m_2w_{21})x_1)}dx_1)dx_2|\\
&\le \int_{\mathbb{T}}|\int_{\mathbb{T}} e^{2\pi i q(h_{\boldsymbol{m}}(x_1,x_2)+(m_1w_{11}+m_2w_{21})x_1)}dx_1|dx_2\\
&\le \frac{1}{2\pi|q|}\int_{\mathbb{T}}|\int_{\mathbb{T}}\frac{1}{\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{11}+m_2w_{21}}de^{2\pi i q(h_{\boldsymbol{m}}(\boldsymbol{x})+(m_1w_{11}+m_2w_{21})x_1)}|dx_2\\
&= \frac{1}{2\pi|q|}\int_{\mathbb{T}}|\int_{\mathbb{T}} e^{2\pi i q(h_{\boldsymbol{m}}(\boldsymbol{x})+(m_1w_{11}+m_2w_{21})x_1)}d\frac{1}{\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{11}+m_2w_{21}}|dx_2\\
&\le \frac{1}{2\pi|q|}\int_{\mathbb{T}}\mathrm{Var}_I\frac{1}{\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\cdot,x_2)+m_1w_{11}+m_2w_{21}}dx_2\\
&\le \frac{2}{\pi|q|}\int_{\mathbb{T}}\frac{\mathrm{Var}_I\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\cdot,x_2)}{|m_1w_{11}+m_2w_{21}|^2}dx_2\\
&\le \frac{2}{\pi|q|}\frac{|m_1|+|m_2|}{|m_1w_{11}+m_2w_{21}|^2}(\|\frac{\partial h_1}{\partial x_1}\|_{BV}+\|\frac{\partial h_2}{\partial x_1}\|_{BV}),
\end{aligned}$$

and finally

$$s_{\boldsymbol{m},q}\le\frac{R}{|m_1w_{11}+m_2w_{21}|},$$

where $R=\frac{4C}{\pi|q|}(\|\frac{\partial h_1}{\partial x_1}\|_{BV}+\|\frac{\partial h_2}{\partial x_1}\|_{BV})$. ■

For a given $s\in\mathbb{N}$ let us denote by V_{1s} the set

$$\{\boldsymbol{m}\in\mathbb{Z}^2\backslash\{0\}; s|m_1w_{12}+m_2w_{22}|\le|m_1w_{11}+m_2w_{21}|\le(s+1)|m_1w_{12}+m_2w_{22}|\}.$$

Then $V_1=\bigcup_{s=1}^{\infty}V_{1s}$.

Lemma 5.4. *There exists a constant $R_*>0$ such that if $\boldsymbol{m}\in V_{1s}$ and*

$$(4Ms)^{1/(1-a)}\le|m_1w_{11}+m_2w_{21}|, \tag{11}$$

then

$$s_{\boldsymbol{m},q}\le\frac{R_*}{|m_1w_{11}+m_2w_{21}|^{3a-1}}.$$

Proof. Analysis similar to that in the proof of Lemma 5.3 shows that for any $\boldsymbol{x}\in\mathbb{T}^2$,

$$\begin{aligned}
\frac{|\frac{\partial}{\partial x_2}h_{\boldsymbol{m}}(\boldsymbol{x})|}{|m_1w_{12}+m_2w_{22}|} &\le M\frac{|m_1w_{11}+m_2w_{21}|^a}{|m_1w_{12}+m_2w_{22}|}\\
&\le \frac{2sM}{|m_1w_{11}+m_2w_{21}|^{1-a}}\le\frac{1}{2}.
\end{aligned}$$

Therefore

$$|\frac{\partial}{\partial x_2}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{12}+m_2w_{22}|\ge\frac{1}{2}|m_1w_{12}+m_2w_{22}|.$$

From the above inequality and (10) we have

$$|\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{11}+m_2w_{21}||\frac{\partial}{\partial x_2}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{12}+m_2w_{22}|\ge$$
$$\frac{1}{4}|m_1w_{11}+m_2w_{21}||m_1w_{12}+m_2w_{22}|.$$

It follows that

$$|qDh_{\boldsymbol{m}}(\boldsymbol{x})-2\pi iq^2(\frac{\partial}{\partial x_1}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{11}+m_2w_{21})(\frac{\partial}{\partial x_2}h_{\boldsymbol{m}}(\boldsymbol{x})+m_1w_{12}+m_2w_{22})|\ge$$
$$q^2|m_1w_{11}+m_2w_{21}||m_1w_{12}+m_2w_{22}|.$$

Applying Lemma 3.6 to the function

$$q(h_{\boldsymbol{m}}(x_1,x_2)+(m_1w_{11}+m_2w_{21})x_1+(m_1w_{12}+m_2w_{22})x_2)$$

we get

$$\begin{aligned}
&s_{\boldsymbol{m},q}\\
&\le \frac{\|Dh_{\boldsymbol{m}}\|_{BV}+2\|\frac{\partial h_{\boldsymbol{m}}}{\partial x_1}\|_{BV}\|\frac{\partial h_{\boldsymbol{m}}}{\partial x_2}\|_{BV}}{q^2|m_1w_{11}+m_2w_{21}|^2|m_1w_{12}+m_2w_{22}|^2}\\
&\quad+\frac{\|Dh_{\boldsymbol{m}}\|_{BV}^2+16\pi\|\frac{\partial h_{\boldsymbol{m}}}{\partial x_1}\|_{BV}^2\|\frac{\partial h_{\boldsymbol{m}}}{\partial x_2}\|_{BV}^2}{q^2|m_1w_{11}+m_2w_{21}|^3|m_1w_{12}+m_2w_{22}|^3}\\
&\le \frac{|m_1|\|Dh_1\|_{BV}+|m_2|\|Dh_2\|_{BV}}{q^2|m_1w_{11}+m_2w_{21}|^2|m_1w_{12}+m_2w_{22}|^2}\\
&\quad+\frac{2(|m_1|\|\frac{\partial h_1}{\partial x_1}\|_{BV}+|m_2|\|\frac{\partial h_2}{\partial x_1}\|_{BV})(|m_1|\|\frac{\partial h_1}{\partial x_2}\|_{BV}+|m_2|\|\frac{\partial h_2}{\partial x_2}\|_{BV})}{q^2|m_1w_{11}+m_2w_{21}|^2|m_1w_{12}+m_2w_{22}|^2}\\
&\quad+\frac{(|m_1|\|Dh_1\|_{BV}+|m_2|\|Dh_2\|_{BV})^2}{q^2|m_1w_{11}+m_2w_{21}|^3|m_1w_{12}+m_2w_{22}|^3}\\
&\quad+\frac{16\pi(|m_1|\|\frac{\partial h_1}{\partial x_1}\|_{BV}+|m_2|\|\frac{\partial h_2}{\partial x_1}\|_{BV})^2(|m_1|\|\frac{\partial h_1}{\partial x_2}\|_{BV}+|m_2|\|\frac{\partial h_2}{\partial x_2}\|_{BV})^2}{q^2|m_1w_{11}+m_2w_{21}|^3|m_1w_{12}+m_2w_{22}|^3}\\
&\le R_1\frac{|m_1|^2+|m_2|^2}{|m_1w_{11}+m_2w_{21}|^2|m_1w_{12}+m_2w_{22}|^2}\\
&\quad+R_2\frac{(|m_1|^2+|m_2|^2)^2}{|m_1w_{11}+m_2w_{21}|^3|m_1w_{12}+m_2w_{22}|^3},
\end{aligned}$$

where

$$R_1=\frac{1}{q^2}(\|Dh_1\|_{BV}+\|Dh_2\|_{BV}+2\sqrt{\|\frac{\partial h_1}{\partial x_1}\|_{BV}^2+\|\frac{\partial h_2}{\partial x_1}\|_{BV}^2}\sqrt{\|\frac{\partial h_1}{\partial x_2}\|_{BV}^2+\|\frac{\partial h_2}{\partial x_2}\|_{BV}^2}$$

and

$$R_2=\frac{1}{q^2}((\|Dh_1\|_{BV}+\|Dh_2\|_{BV})^2+16\pi(\|\frac{\partial h_1}{\partial x_1}\|_{BV}^2+\|\frac{\partial h_2}{\partial x_1}\|_{BV}^2)(\|\frac{\partial h_1}{\partial x_2}\|_{BV}^2+\|\frac{\partial h_2}{\partial x_2}\|_{BV}^2)).$$

By Lemma 1.5,

$$\begin{aligned}
s_{\boldsymbol{m},q} &\le \frac{2R_1C^2}{|m_1w_{12}+m_2w_{22}|^2}+\frac{4R_2C^4|m_1w_{11}+m_2w_{21}|}{|m_1w_{12}+m_2w_{22}|^3}\\
&\le \frac{R_*|m_1w_{11}+m_2w_{21}|}{|m_1w_{12}+m_2w_{22}|^3},
\end{aligned}$$

where $R_* = 2R_1C^2 + 4R_2C^4$. From (11) we have

$$\begin{aligned}\frac{1}{|m_1w_{12}+m_2w_{22}|} &\leq \frac{2s}{|m_1w_{11}+m_2w_{21}|} \leq \frac{|m_1w_{11}+m_2w_{21}|^{1-a}}{2M|m_1w_{11}+m_2w_{21}|} \\ &\leq \frac{1}{|m_1w_{11}+m_2w_{21}|^a}.\end{aligned}$$

Therefore

$$s_{\boldsymbol{m},q} \leq \frac{R_*}{|m_1w_{11}+m_2w_{21}|^{3a-1}}. \blacksquare$$

Proof of Theorem 5.2. Set $U_1 = \bigcup_{s=1}^{\infty}\{\boldsymbol{m} \in V_{1s}; |m_1w_{11}+m_2w_{21}| \geq (4Ms)^{1/(1-a)}\}$. The set $\{\boldsymbol{m} \in U_1; |m_1w_{11}+m_2w_{21}| = k\}$ has at most $2(2k+1)$ members. By Lemma 5.4, we have

$$\sum_{\boldsymbol{m}\in U_1} s^2_{\boldsymbol{m},q} \leq \sum_{\boldsymbol{m}\in U_1} \frac{R_*^2}{|m_1w_{11}+m_2w_{21}|^{6a-2}} \leq \sum_{k=1}^{\infty} \frac{5R_*^2k}{k^{6a-2}} \leq 5R_*^2 \sum_{k=1}^{\infty}\frac{1}{k^{6a-3}} < \infty,$$

because $6a-3 > 3/2$. Set $Z_{1s} = \{\boldsymbol{m} \in V_{1s}; (2M)^{1/(1-a)} \leq |m_1w_{11}+m_2w_{21}| \leq (4Ms)^{1/(1-a)}\}$ and $Z_1 = \bigcup_{s=1}^{\infty} Z_{1s}$. The set $\{\boldsymbol{m} \in V_{1s}; |m_1w_{11}+m_2w_{21}| = k\}$ has at most $4k/s(s+1)$ members. By Lemma 5.3,

$$\begin{aligned}\sum_{\boldsymbol{m}\in Z_{1s}} s^2_{\boldsymbol{m},q} &\leq \sum_{\boldsymbol{m}\in Z_{1s}} \frac{R^2}{|m_1w_{11}+m_2w_{21}|^2} \\ &\leq R^2 \sum_{k=1}^{[(4Ms)^{1/(1-a)}]} \frac{1}{k^2}\frac{4k}{s(s+1)} \\ &\leq 8R^2\frac{\ln(4Ms)^{1/(1-a)}}{s(s+1)} \\ &= \frac{8R^2}{1-a}\frac{\ln(4Ms)}{s(s+1)}.\end{aligned}$$

It follows that

$$\sum_{\boldsymbol{m}\in Z_1} s^2_{\boldsymbol{m},q} \leq \frac{8R^2}{1-a}\sum_{s=1}^{\infty}\frac{\ln(4Ms)}{s(s+1)} < \infty.$$

Since the set $V_1 \setminus (Z_1 \cup U_1) = \{\boldsymbol{m} \in V_1; |m_1w_{11}+m_2w_{21}| < (2M)^{1/(1-a)}\}$ is finite, we obtain that $\sum_{\boldsymbol{m}\in V_1} s^2_{\boldsymbol{m},q} < \infty$, which completes the proof of the theorem.

$\blacksquare$

Corollary 5.1. *If $\boldsymbol{T}$ is of finite type and ϕ is of class C^4, then $\boldsymbol{T}_\phi$ has countable Lebesgue spectrum on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.*

Unfortunately, one cannot prove Theorem 5.2 in the case $d > 2$. For $d \geq 2$ we show that $\boldsymbol{T}_\phi$ has countable Lebesgue spectrum in the simplest case when ϕ is an affine cocycle, i.e. the functions $h_1, ..., h_d$ are constant.

Theorem 5.5. *Let $\boldsymbol{T}$ be an ergodic and free $\mathbb{Z}^d$–rotation on $\mathbb{T}^d$. If $\phi : \mathbb{Z}^d \times \mathbb{T}^d \to \mathbb{T}$ is an affine cocycle and $\det W(\phi) \neq 0$, then $\boldsymbol{T}_\phi$ has countable Lebesgue spectrum on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.*

Proof. For any $q \in \mathbb{Z} \setminus \{0\}$ and $\boldsymbol{m} \in \mathbb{Z}^d$ we have

$$s_{\boldsymbol{m},q} = \left| \int_{\mathbb{T}^d} e^{2\pi i q\, \boldsymbol{m}\, W(\phi)\, \boldsymbol{x}^T} d\,\boldsymbol{x} \right|.$$

Since $W(\phi)^T : \mathbb{Z}^d \to \mathbb{Z}^d$ is monomorphism, we obtain that $s_{\boldsymbol{m},q} = 0$ for $\boldsymbol{m} \neq 0$ and finally that $\sum_{\boldsymbol{m}\in\mathbb{Z}^d} s^2_{\boldsymbol{m},q} < \infty$. ■

6 The case $\det W(\phi) = 0$

Given an irrational number $\alpha \in [0,1)$, let $[0; a_1, a_2, ...]$ be its continued fraction expansion, where a_n are positive integers. Put

$$q_0 = 1,\ q_1 = a_1,\ q_{n+1} = a_{n+1}q_n + q_{n-1},$$

$$p_0 = 0,\ p_1 = 1,\ p_{n+1} = a_{n+1}p_n + p_{n-1}.$$

The rationals p_n/q_n are called the *convergents* of α.

Lemma 6.1. *(See [4]) If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is of bounded variation, then for any $x \in \mathbb{R}$*

$$|f(x) + f(x+\alpha) + ... + f(x + (q_n - 1)\alpha) - q_n \int_I f(x)dx| \leq \mathrm{Var} f. \tag{12}$$

If $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is absolutely continuous, then

$$f(\cdot) + f(\cdot + \alpha) + ... + f(\cdot + (q_n - 1)\alpha) - q_n \int_{\mathbb{T}} f(x)dx$$

converges uniformly to zero. ■

Lemma 6.2. *Let $\boldsymbol{T} : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ be an ergodic and free $\mathbb{Z}^2$–rotation given by*

$$\boldsymbol{T}_{(m_1,m_2)}(z_1, z_2) = (e^{2\pi i(\alpha_{11}m_1 + \alpha_{12}m_2)} z_1, e^{2\pi i(\alpha_{21}m_1 + \alpha_{22}m_2)} z_2)$$

such that the automorphism $\boldsymbol{T}_{(k_1,k_2)}$ is not ergodic for some $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$. If $\phi : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}$ is an absolutely continuous $\mathbb{T}$–cocycle such that

$$k_1 w_{11} + k_2 w_{21} = k_1 w_{12} + k_2 w_{22} = 0,$$

then the maximal spectral type of $\boldsymbol{T}_\phi$ is singular and $\boldsymbol{T}_\phi$ is not mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.

Proof. Set

$$\begin{aligned} c_{\boldsymbol{m},r} &= \int_{\mathbb{T}^2} \phi_{\boldsymbol{m}}(\boldsymbol{z})^r d\,\boldsymbol{z} \\ &= \int_{\mathbb{T}^2} e^{2\pi i r(h_{\boldsymbol{m}}(x_1,x_2) + (m_1 w_{11} + m_2 w_{21})x_1 + (m_1 w_{12} + m_2 w_{22})x_2)} dx_1 dx_2. \end{aligned}$$

In view of Lemma 1.2 and Lemma 1.3 it is enough to find a sequence $\{\boldsymbol{m}^{(n)}\}_{n=1}^{\infty}$ in $\mathbb{Z}^2$ ($\boldsymbol{m}^{(n)} \to \infty$, as $n \to \infty$) and a real number $c \in \mathbb{R}$ such that for any $r \in \mathbb{Z} \setminus \{0\}$ we have

$$\lim_{n\to\infty} c_{\boldsymbol{m}^{(n)},r} = e^{2\pi irc}.$$

Put $\alpha_1 = \alpha_{11}k_1 + \alpha_{12}k_2$ and $\alpha_2 = \alpha_{21}k_1 + \alpha_{22}k_2$. Since the rotation $\boldsymbol{T}_{(k_1,k_2)}$ is not ergodic, there exist integers l_1, l_2, l_3, ($l_1^2 + l_2^2 \neq 0$) such that $l_1\alpha_1 + l_2\alpha_2 = l_3$. Take $s_1, s_2 \in \mathbb{Z}$ such that $s_1 l_2 - s_2 l_1 = \gcd(l_1, l_2)$. Put $l = \gcd(l_1, l_2)$ and $\alpha = s_1\alpha_1 + s_2\alpha_2$. Let p_n/q_n be the convergents of $l\alpha$. Set

$$B = \begin{bmatrix} s_1 & s_2 \\ l_1/l & l_2/l \end{bmatrix}.$$

Then $B \in M_2(\mathbb{Z})$ and $\det B = 1$. Consider the linear operator $P : L^1(\mathbb{R}^2/\mathbb{Z}^2) \to L^1(\mathbb{R}^2/\mathbb{Z}^2)$ given by

$$Pf(x_1, x_2) = \int_{\mathbb{T}} fB^{-1}(x, \frac{l_1x_1 + l_2x_2}{l})dx.$$

Let $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}$ be an absolutely continuous function. Then the function $fB^{-1}(\cdot, y) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is absolutely continuous for any $y \in \mathbb{R}$. It follows that

$$\begin{aligned} &\sum_{j=0}^{q_n-1} f(x_1 + jl\alpha_1, x_2 + jl\alpha_2) - q_n Pf(x_1, x_2) \\ &\qquad = \sum_{j=0}^{q_n-1} fB^{-1}(y_1 + jl\alpha, y_2) - q_n \int_{\mathbb{T}} fB^{-1}(x, y_2)dx, \end{aligned}$$

where $y_1 = s_1x_1 + s_2x_2$ and $y_2 = (l_1x_1 + l_2x_2)/l$. By Lemma 6.1, for any $(x_1, x_2) \in \mathbb{R}^2$ we have

$$\lim_{n\to\infty} \sum_{j=0}^{q_n-1} f(x_1 + jl\alpha_1, x_2 + jl\alpha_2) - q_n Pf(x_1, x_2) = 0.$$

Applying inequality (12) and Lemma 3.1 we get

$$|\sum_{j=0}^{q_n-1} f(x_1 + jl\alpha_1, x_2 + jl\alpha_2) - q_n Pf(x_1, x_2)| \leq \mathrm{Var}_I fB^{-1}(\cdot, y_2) \leq C\|f\|_{BV}.$$

By Lebesgue's bounded convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{T}^2} |\sum_{j=0}^{q_n-1} f(x_1 + jl\alpha_1, x_2 + jl\alpha_2) - q_n Pf(x_1, x_2)|dx_1dx_2 = 0. \tag{13}$$

Since for any natural m we have

$$h_{(mlk_1, mlk_2)}(x_1, x_2) = \sum_{j=0}^{m-1} h_{(lk_1, lk_2)}(x_1 + jl\alpha_1, x_2 + jl\alpha_2),$$

the sequence $\{h_{(q_n lk_1, q_n lk_2)} - q_n P h_{(lk_1,lk_2)}\}_{n\in\mathbb{N}}$ converges to zero in $L^1(\mathbb{R}^2/\mathbb{Z}^2)$, by (13). Observe that $P = P_{(l\alpha_1, l\alpha_2)}$. By Lemma 2.1,

$$Ph_{(lk_1,lk_2)} = \int_{\mathbb{T}^2} h_{(lk_1,lk_2)} d\lambda.$$

Put $\boldsymbol{m}^{(n)} = (q_n lk_1, q_n lk_2)$. Then

$$h_{\boldsymbol{m}^{(n)}} - q_n \int_{\mathbb{T}^2} h_{(lk_1,lk_2)} d\lambda$$

converges to zero in $L^1(\mathbb{R}^2/\mathbb{Z}^2)$. Without loss of generality we can assume that

$$\lim_{n\to\infty} e^{2\pi i q_n \int_{\mathbb{T}^2} h_{(lk_1,lk_2)} d\lambda} = e^{2\pi i c}.$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{T}^2} \phi_{\boldsymbol{m}^{(n)}}(\boldsymbol{z})^r d\,\boldsymbol{z} &= \lim_{n\to\infty} \int_{\mathbb{T}^2} e^{2\pi i r h_{\boldsymbol{m}^{(n)}}(x_1,x_2)} \\ &= \lim_{n\to\infty} e^{2\pi i r q_n \int_{\mathbb{T}^2} h_{(lk_1,lk_2)} d\lambda} = e^{2\pi i r c} \end{aligned}$$

for any $r \in \mathbb{Z} \setminus \{0\}$, which proves the lemma. ■

By the above lemma we have proved the following:

Theorem 6.3. *Let $\boldsymbol{T} : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ be an ergodic and free $\mathbb{Z}^2$–rotation and let $\phi : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}$ be an absolutely continuous $\mathbb{T}$–cocycle with $W(\phi) = 0$. If the automorphism $\boldsymbol{T}_{\boldsymbol{m}}$ is not ergodic for some $\boldsymbol{m} \in \mathbb{Z}^2 \setminus \{0\}$, then the maximal spectral type of $\boldsymbol{T}_{\phi}$ is singular and $\boldsymbol{T}_{\phi}$ is not mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.*

Theorem 6.4. *Let $\boldsymbol{T} : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ be an ergodic and free $\mathbb{Z}^2$–rotation. If $\phi : \mathbb{Z}^2 \times \mathbb{T}^2 \to \mathbb{T}$ is an absolutely continuous $\mathbb{T}$–cocycle and* rank $W(\phi) = 1$, *then the maximal spectral type of $\boldsymbol{T}_{\phi}$ is singular and $\boldsymbol{T}_{\phi}$ is not mixing on the orthocomplement of the eigenfunctions of $\boldsymbol{T}$.*

Proof. By (2), $(W\boldsymbol{\alpha}) - (W\boldsymbol{\alpha})^T \in M_2(\mathbb{Z})$ hence

$$w_{11}\alpha_{12} + w_{12}\alpha_{22} - w_{21}\alpha_{11} - w_{22}\alpha_{21} = d \in \mathbb{Z}.$$

Since $W(\phi) \neq 0$, at least one of the pairs $(w_{21}, -w_{11})$, $(w_{22}, -w_{12})$ is not equal to zero. Assume that $(w_{21}, -w_{11}) \neq 0$. Put $k_1 = w_{21}$ and $k_2 = -w_{11}$. Then $k_1 w_{11} + k_2 w_{21} = 0$. Since $\det W(\phi) = 0$, we have $k_1 w_{12} + k_2 w_{22} = 0$. Set $\alpha_1 = \alpha_{11}k_1 + \alpha_{12}k_2$ and $\alpha_2 = \alpha_{21}k_1 + \alpha_{22}k_2$. Then

$$w_{11}\alpha_1 + w_{12}\alpha_2 = -\alpha_{21} \det W(\phi) - dw_{11} \in \mathbb{Z}$$

and

$$w_{21}\alpha_1 + w_{22}\alpha_2 = -\alpha_{22} \det W(\phi) - dw_{21} \in \mathbb{Z}.$$

Therefore the rotation $\boldsymbol{T}_{(k_1,k_2)}$ is not ergodic, because at least one of the pairs (w_{11}, w_{12}), (w_{21}, w_{22}) is not equal to zero. It follows that the rotation $\boldsymbol{T}$ and the cocycle ϕ satisfy the assumptions of Lemma 6.2, and the proof is complete. ■

Our considerations overlook the case when all rotations $\boldsymbol{T}_{\boldsymbol{m}}$ are ergodic and $W(\phi) = 0$. It would be interesting to answer the question: what kind of spectra can be obtained in this case?

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