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Measure-preserving diffeomorphisms of the torus

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Abstract. We consider measure-preserving diffeomorphisms of the two-dimensional torus with zero entropy. We prove that every ergodic C^3-diffeomorphism f of the two-dimensional torus with linear growth of the derivative (i.e. the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is uniformly separated from 0 and ∞ and it is bounded in the C^2-norm) is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^3-cocycle with non-zero topological degree.

1. *Introduction*

Let M be a compact Riemannian smooth manifold and μ its probability Lebesgue measure. Let $f : (M, \mu) \to (M, \mu)$ be a smooth measure-preserving ergodic diffeomorphism. An important question of smooth ergodic theory is: what is the relation between the asymptotic properties of the sequence $\{Df^n\}_{n\in\mathbb{N}}$ and the dynamical or spectral properties of the dynamical system $f : (M, \mu) \to (M, \mu)$. There are results which describe this relation well in the case where M is the torus. For example, if a diffeomorphism f is homotopic to the identity and the sequence $\{Df^n\}_{n\in\mathbb{N}}$ is uniformly bounded, then f is C^0-conjugate to an ergodic rotation (see [**2**, p. 181]). Hence f has a purely discrete spectrum. Moreover, if $\{Df^n\}_{n\in\mathbb{N}}$ is bounded in the C^r-norm ($r \in \mathbb{N} \cup \{\infty\}$), then f and the ergodic rotation are C^r-conjugated (see [**2**, p. 181]). However, if $\{\|Df^n\|\}_{n\in\mathbb{N}}$ has ‘exponential growth’, precisely if f is an Anosov diffeomorphism, then it is metrically isomorphic to a Bernoulli shift (see [**5**]). Hence f has a countable Lebesgue spectrum. Moreover, f is C^0-conjugate to an algebraic automorphism of the torus (see [**4**]).

The aim of this paper is to explain what can happen between these extreme cases. Precisely, we study the properties of measure-preserving diffeomorphisms f of the two-dimensional torus for which the sequence $\{Df^n\}_{n\in\mathbb{N}}$ has linear growth. One definition of the linear growth of the derivative is presented in [**1**]. In this paper, it is proved that if the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ converges μ-a.e. to a measurable μ-non-zero function, then f is algebraically conjugate (i.e. by a group automorphism) to a skew product of an irrational rotation on the circle and a circle smooth cocycle with non-zero topological degree.

Moreover, every skew product of an irrational rotation on the circle and a circle C^2-cocycle with non-zero degree has countable Lebesgue spectrum on the orthocomplement of the space of functions depending only on the first variable (see [**3**]). It follows that every measure-preserving, ergodic diffeomorphism with the previously mentioned linear growth of the derivative has countable Lebesgue spectrum on the orthocomplement of its eigenfunctions.

In this paper we propose a seemingly weaker definition of the linear growth of the derivative.

2. *Notations, definition and basic remarks*

By $\mathbb{T}^2$ ($\mathbb{T}$ respectively) we will mean the torus $\mathbb{R}^2/\mathbb{Z}^2$ (the circle $\mathbb{R}/\mathbb{Z}$ respectively); by λ will denote Lebesgue measure on $\mathbb{T}^2$. We will identify functions on $\mathbb{T}^2$ with $\mathbb{Z}^2$-periodic functions (i.e. periodic of period 1 in each coordinates) on $\mathbb{R}^2$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a smooth diffeomorphism. We will identify f with a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(x_1+1, x_2) = f(x_1, x_2) + (a_{11}, a_{21}),$$
$$f(x_1, x_2+1) = f(x_1, x_2) + (a_{12}, a_{22})$$

for every $(x_1, x_2) \in \mathbb{R}^2$, where $[a_{ij}]_{i,j=1,2} \in GL_2(\mathbb{Z})$. Then there exist smooth functions $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{T}^2 \to \mathbb{R}$ such that

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2), a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)).$$

We will denote by $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ the coordinate functions of f. By $M_2(\mathbb{R})$ we mean the space 2×2 matrices endowed with the operator norm.

Definition 1. We say that the derivative of a smooth diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ has *linear growth* if there exist positive constants c, C such that

$$0 < c \le \frac{1}{n}\|Df^n(\bar{x})\| \le C \tag{1}$$

for every $\bar{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}$.

One of the examples of ergodic measure-preserving diffeomorphisms with linear growth of the derivative is any skew product of any irrational rotation on the circle and any circle smooth cocycle with non-zero degree. Let $\alpha \in \mathbb{T}$ be an irrational number and let $\varphi : \mathbb{T} \to \mathbb{T}$ be a C^1-cocycle. We denote by $d(\varphi)$ the topological degree of φ. Consider the skew product $T_{\alpha,\varphi} : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ defined by

$$T_{\alpha,\varphi}(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)).$$

LEMMA 1. *The sequence $n^{-1}DT^n_{\alpha,\varphi}$ converges uniformly to the matrix* $\begin{bmatrix} 0 & 0 \\ d(\varphi) & 0 \end{bmatrix}$.

Proof. Observe that

$$\frac{1}{n}DT^n_{\alpha,\varphi}(x_1, x_2) = \begin{bmatrix} \frac{1}{n} & 0 \\ \frac{1}{n}\sum_{k=0}^{n-1} D\varphi(x_1 + k\alpha) & \frac{1}{n} \end{bmatrix}.$$

By the ergodic theorem, the sequence $n^{-1}\sum_{k=0}^{n-1} D\varphi(\,\cdot\, + k\alpha)$ converges uniformly to the number $\int_{\mathbb{T}} D\varphi(x)\,dx = d(\varphi)$. □

It follows that if $d(\varphi) \neq 0$, then $T_{\alpha,\varphi}$ has linear growth of the derivative. Let $r \in \mathbb{N}$. It is easy to check that if φ is of class C^r, then

$$\max_{1\le i\le r}\sup_{n\in\mathbb{N}}\sup_{\bar{x}\in\mathbb{T}^2}\frac{1}{n}\|D^i T^n_{\alpha,\varphi}(\bar{x})\| < \infty.$$

Our definition has a nice property because the linear growth of the derivative is invariant under the relation of smooth conjugation. Indeed, suppose that two C^r-diffeomorphisms f_1 and f_2 of $\mathbb{T}^2$ are C^r-conjugated, i.e. there exists C^r-diffeomorphism $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$f_1 \circ \psi = \psi \circ f_2.$$

Then

$$Df_1^n \circ \psi = D\psi \circ f_2^n \cdot Df_2^n \cdot D\psi^{-1} \circ \psi$$

and

$$Df_2^n = D\psi^{-1} \circ \psi \circ f_2^n \cdot Df_1^n \circ \psi \cdot D\psi$$

for any natural n. Therefore

$$K^{-1}\|Df_2^n(\bar{x})\| \le \|Df_1^n(\psi\bar{x})\| \le K\|Df_2^n(\bar{x})\|$$

for every $\bar{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}$, where

$$K = \sup_{\bar{x}\in\mathbb{T}^2}\|D\psi(\bar{x})\| \cdot \sup_{\bar{x}\in\mathbb{T}^2}\|D\psi^{-1}(\bar{x})\|.$$

It follows that if

$$0 < c \le \frac{1}{n}\|Df_1^n(\bar{x})\| \le C,$$

then

$$0 < c/K \le \frac{1}{n}\|Df_2^n(\bar{x})\| \le CK$$

for every $\bar{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}$. Moreover, if $\psi : \mathbb{T}^2 \to \mathbb{T}^2$ is a group automorphism, then

$$D^i f_1^n(\psi\bar{x}) \cdot (D\psi(\bar{x}))^i = D\psi(\bar{x}) \cdot D^i f_2^n(\bar{x})$$

for any $\bar{x} \in \mathbb{T}^2$ and $1 \le i \le r$. Therefore there exists $M > 0$ such that

$$M^{-1}\sup_{1\le i\le r}\|D^i f_2^n(\bar{x})\| \le \sup_{1\le i\le r}\|D^i f_1^n(\psi\bar{x})\| \le M\sup_{1\le i\le r}\|D^i f_2^n(\bar{x})\|$$

for every $\bar{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}$.

Let $\langle B, \|\cdot\|\rangle$ be a Banach space and let $r \in \mathbb{N} \cup \{0\}$. We will denote by $C^k(\mathbb{T}^2, B)$ the space C^k-functions $f : \mathbb{T}^2 \to B$ endowed with the norm given by

$$\|f\|_r = \max_{0\le i\le r}\sup_{\bar{x}\in\mathbb{T}^2}\|D^i f(\bar{x})\|.$$

From this, we reach the following conclusion.

COROLLARY 2. *If a measure-preserving C^3-diffeomorphism $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^3-cocycle with non-zero degree, then:*

- *f is ergodic;*
- *f has linear growth of the derivative; and*
- *the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$.*

In this paper we will prove the converse of Corollary 2.

THEOREM 3. *(Main theorem) Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^3-diffeomorphism. Suppose that:*

- *f is ergodic;*
- *f has linear growth of the derivative; and*
- *the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$.*

Then f is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^3-cocycle with non-zero degree.

In addition, our theorem leads to the following conclusion. If f is ergodic, has linear growth of the derivative and the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in the C^2-norm, then f has a countable Lebesgue spectrum on the orthocomplement of its eigenfunctions.

3. *General remarks about the linear growth*

Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^3-diffeomorphism. Assume that f has linear growth of the derivative, i.e. satisfies (1). In this section it is shown that there is something like an 'unstable' and a 'stable' direction for f at each point. A direction $u(\bar{x}) \in S^1$ is 'unstable' if

$$\lim_{n\to\infty} \frac{1}{n}(\|Df^n(\bar{x})u(\bar{x})\| - \|Df^n(\bar{x})\|) = 0 \tag{2}$$

and a direction $v(\bar{x}) \in S^1$ is 'stable' if

$$\lim_{n\to\infty} \frac{1}{n}\|Df^n(\bar{x})v(\bar{x})\| = 0. \tag{3}$$

Moreover, if the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$, then u and v can be chosen in a smooth way and they are unique up to ± 1.

Fix $\bar{x} \in \mathbb{T}^2$ and $n \in \mathbb{N}$. Set $B_n(\bar{x}) = Df^n(\bar{x})$. Let $A_n(\bar{x}) \in M_2(\mathbb{R})$ be a (positive) symmetric matrix such that $A_n(\bar{x})^2 = B_n(\bar{x})^T B_n(\bar{x})$. Let $\lambda_n(\bar{x}) > \mu_n(\bar{x}) > 0$ be eigenvalues of $A_n(\bar{x})$. Then $\lambda_n(\bar{x})\mu_n(\bar{x}) = 1$ and $\lambda_n(\bar{x}) = \|A_n(\bar{x})\| = \|B_n(\bar{x})\|$. Hence $nc \le \lambda_n(\bar{x}) \le nC$. Let $u_n(\bar{x})$ and $v_n(\bar{x})$ be the normalized eigenvectors of $A_n(\bar{x})$ with eigenvalues $\lambda_n(\bar{x})$ and $\mu_n(\bar{x})$. Then $u_n(\bar{x})$ and $v_n(\bar{x})$ are perpendicular.

LEMMA 4. *If $\langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle \ge 0$ for $n \ge n_0$, then $\lim_{n\to\infty} u_n(\bar{x}) = u(\bar{x})$. Moreover, there exists $K > 0$ independent of $\bar{x}$ and n_0 such that $\|u_n(\bar{x}) - u(\bar{x})\| \le K/n$ for $n \ge n_0$ and*

$$\lim_{n\to\infty} \frac{1}{n}(\|Df^n(\bar{x})u(\bar{x})\| - \|Df^n(\bar{x})\|) = 0.$$

If $\langle v_n(\bar{x}), v_{n+1}(\bar{x})\rangle \geq 0$ *for* $n \geq n_0$, *then* $\lim_{n\to\infty} v_n(\bar{x}) = v(\bar{x})$. *Moreover,* $\|v_n(\bar{x}) - v(\bar{x})\| \leq K/n$ *for* $n \geq n_0$ *and*

$$\lim_{n\to\infty} \frac{1}{n}\|Df^n(\bar{x})v(\bar{x})\| = 0.$$

Proof. Since $0 \leq \langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle \leq 1$, we have

$$\|u_n(\bar{x}) - u_{n+1}(\bar{x})\|^2 = 2(1 - \langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle) \leq 2(1 - \langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle^2).$$

On the other hand,

$$1 = \|u_n(\bar{x})\|^2 = \langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle^2 + \langle u_n(\bar{x}), v_{n+1}(\bar{x})\rangle^2.$$

Hence

$$\|u_n(\bar{x}) - u_{n+1}(\bar{x})\| \leq \sqrt{2}|\langle u_n(\bar{x}), v_{n+1}(\bar{x})\rangle|.$$

However,

$$\begin{aligned}
|\langle u_n(\bar{x}), v_{n+1}(\bar{x})\rangle| &= \frac{1}{\lambda_n(\bar{x})}|\langle A_n(\bar{x})u_n(\bar{x}), v_{n+1}(\bar{x})\rangle| \\
&= \frac{1}{\lambda_n(\bar{x})}|\langle u_n(\bar{x}), A_n(\bar{x})v_{n+1}(\bar{x})\rangle| \\
&\leq \frac{1}{\lambda_n(\bar{x})}\|B_n(\bar{x})v_{n+1}(\bar{x})\| \\
&= \frac{1}{\lambda_n(\bar{x})}\|Df^{-1}(f^{n+1}\bar{x})Df^{n+1}(\bar{x})v_{n+1}(\bar{x})\| \\
&\leq \frac{1}{\lambda_n(\bar{x})}\sup_{\bar{y}\in\mathbb{T}^2}\|Df^{-1}(\bar{y})\|\|A_{n+1}(\bar{x})v_{n+1}(\bar{x})\| \\
&= \frac{1}{\lambda_n(\bar{x})\lambda_{n+1}(\bar{x})}\sup_{\bar{y}\in\mathbb{T}^2}\|Df^{-1}(\bar{y})\|.
\end{aligned}$$

It follows that

$$\|u_n(\bar{x}) - u_{n+1}(\bar{x})\| \leq \frac{K}{n^2}$$

for $n \geq n_0$, where $K = \sqrt{2}\sup_{\bar{y}\in\mathbb{T}^2}\|Df^{-1}(\bar{y})\|/c^2$. Therefore

$$\lim_{n\to\infty} u_n(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \|u_n(\bar{x}) - u(\bar{x})\| \leq K/n$$

for $n \geq n_0$. Similarly, we can prove that

$$\lim_{n\to\infty} v_n(\bar{x}) = v(\bar{x}) \quad \text{and} \quad \|v_n(\bar{x}) - v(\bar{x})\| \leq K/n$$

for $n \geq n_0$. Moreover,

$$\begin{aligned}
\frac{1}{n}(\|Df^n(\bar{x})u(\bar{x})\| - \|Df^n(\bar{x})\|) &= \frac{1}{n}(\|A_n(\bar{x})u(\bar{x})\| - \lambda_n(\bar{x})) \\
&\leq \frac{1}{n}(\|A_n(\bar{x})(u(\bar{x}) - u_n(\bar{x}))\| + \|A_n(\bar{x})u_n(\bar{x})\| - \lambda_n(\bar{x})) \\
&\leq \frac{1}{n}\|A_n(\bar{x})\|\|u(\bar{x}) - u_n(\bar{x})\| \\
&\leq \frac{CK}{n}
\end{aligned}$$

and

$$\frac{1}{n}\|Df^n(\bar{x})v(\bar{x})\| \le \frac{1}{n}(\|A_n(\bar{x})(v(\bar{x}) - v_n(\bar{x}))\| + \|A_n(\bar{x})v_n(\bar{x})\|)$$
$$\le \frac{1}{n}(\|A_n(\bar{x})\|\|v(\bar{x}) - v_n(\bar{x})\| + \mu_n(\bar{x})) \le \frac{CK}{n} + \frac{1}{cn^2}$$

for $n \ge n_0$. Letting $n \to \infty$, we obtain our claim. □

LEMMA 5. *Let $\bar{x} \in \mathbb{T}^2$ and let $\{n_i\}_{i\in\mathbb{N}}$ be an increasing sequence of natural numbers. Suppose that $u^1, v^1, u^2, v^2 \in S^1$ satisfy*

$$\lim_{i\to\infty} \frac{1}{n_i}(\|Df^{n_i}(\bar{x})u^j\| - \|Df^{n_i}(\bar{x})\|) = \lim_{i\to\infty} \frac{1}{n_i}\|Df^{n_i}(\bar{x})v^j\| = 0$$

for $j = 1, 2$. Assume that $u^1 \perp v^1$. Then $u^2 = \pm u^1$ and $v^2 = \pm v^1$.

Proof. Since $u^1 \perp v^1$, we have

$$\|Df^n(\bar{x})u^2\| = \|\langle u^2, u^1\rangle Df^n(\bar{x})u^1 + \langle u^2, v^1\rangle Df^n(\bar{x})v^1\|$$

and

$$|\langle u^2, v^1\rangle|\|Df^n(\bar{x})v^1\| \ge |\|Df^n(\bar{x})u^2\| - |\langle u^2, u^1\rangle|\|Df^n(\bar{x})u^1\||$$

for all n. It follows that

$$|\langle u^2, v^1\rangle|\frac{1}{n_i}\|Df^{n_i}(\bar{x})v^1\| \ge \Bigg|\frac{1}{n_i}(\|Df^{n_i}(\bar{x})u^2\| - \|Df^{n_i}(\bar{x})\|)$$
$$- |\langle u^2, u^1\rangle|\frac{1}{n_i}(\|Df^{n_i}(\bar{x})u^1\| - \|Df^{n_i}(\bar{x})\|)$$
$$+ (1 - |\langle u^2, u^1\rangle|)\frac{1}{n_i}\|Df^{n_i}(\bar{x})\|\Bigg|$$

for any natural i. Letting $i \to \infty$, we obtain

$$\lim_{i\to\infty}(1 - |\langle u^2, u^1\rangle|)\frac{1}{n_i}\|Df^{n_i}(\bar{x})\| = 0.$$

Since $n^{-1}\|Df^n(\bar{x})\| \ge c > 0$ for any natural n, we conclude that $\langle u^2, u^1\rangle = \pm 1$, hence that $u^2 = \pm u^1$. Similarly we can prove that $v^2 = \pm v^1$. □

LEMMA 6. *Assume that $\sup_{n\in\mathbb{N}} n^{-1}\|Df^n\|_1 = M < \infty$. Then there exist $r > 0$ and $L > 0$ such that for every $\bar{x}_0 \in \mathbb{R}^2$ we can choose $u, v : \mathbb{R}^2 \to S^1$ satisfying (2) and (3) for which the functions $u, v : \{\bar{x} \in \mathbb{R}^2 : \|\bar{x} - \bar{x}_0\| < r\} \to S^1$ are Lipschitz with constant equal L.*

Proof. First, choose sequences $\{u_n(\bar{x}_0)\}_{n\in\mathbb{N}}$ and $\{v_n(\bar{x}_0)\}_{n\in\mathbb{N}}$ with $\langle u_n(\bar{x}_0), u_{n+1}(\bar{x}_0)\rangle \ge 0$ and $\langle v_n(\bar{x}_0), v_{n+1}(\bar{x}_0)\rangle \ge 0$ for every natural n. By Lemma 4, $\lim_{n\to\infty} u_n(\bar{x}_0) = u(\bar{x}_0)$ and $\|u_n(\bar{x}_0) - u(\bar{x}_0)\| \le K/n$.

Let $\bar{x} \in \mathbb{R}^2$. Choose a sequence $\{u_n(\bar{x})\}_{n\in\mathbb{N}}$ for which $\langle u_n(\bar{x}_0), u_n(\bar{x})\rangle \ge 0$ for any natural n. Then

$$|\langle u_n(\bar{x}_0), u_{n+1}(\bar{x}_0)\rangle - \langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle|$$
$$\le |\langle u_n(\bar{x}_0) - u_n(\bar{x}), u_{n+1}(\bar{x}_0)\rangle| + |\langle u_n(\bar{x}), u_{n+1}(\bar{x}_0) - u_{n+1}(\bar{x})\rangle|$$
$$\le \|u_n(\bar{x}_0) - u_n(\bar{x})\| + \|u_{n+1}(\bar{x}_0) - u_{n+1}(\bar{x})\|$$

and

$$\begin{aligned}\|u_n(\bar{x}_0) - u_n(\bar{x})\|^2 &= 2(1 - \langle u_n(\bar{x}_0), u_n(\bar{x})\rangle) \\ &\le 2(1 - \langle u_n(\bar{x}_0), u_n(\bar{x})\rangle^2) \\ &= 2\langle u_n(\bar{x}_0), v_n(\bar{x})\rangle^2\end{aligned}$$

because

$$1 = \|u_n(\bar{x}_0)\|^2 = \langle u_n(\bar{x}_0), u_n(\bar{x})\rangle^2 + \langle u_n(\bar{x}_0), v_n(\bar{x})\rangle^2.$$

Therefore

$$\begin{aligned}\frac{1}{\sqrt{2}}\|u_n(\bar{x}_0) - u_n(\bar{x})\| &= |\langle u_n(\bar{x}_0), v_n(\bar{x})\rangle| \\ &= \frac{1}{\lambda_n(\bar{x}_0)}|\langle A_n(\bar{x}_0)u_n(\bar{x}_0), v_n(\bar{x})\rangle| \\ &= \frac{1}{\lambda_n(\bar{x}_0)}|\langle u_n(\bar{x}_0), A_n(\bar{x}_0)v_n(\bar{x})\rangle| \\ &\le \frac{1}{\lambda_n(\bar{x}_0)}\|A_n(\bar{x}_0)v_n(\bar{x})\| \\ &= \frac{1}{\lambda_n(\bar{x}_0)}\|Df^n(\bar{x}_0)v_n(\bar{x})\| \\ &\le \frac{1}{\lambda_n(\bar{x}_0)}(\|(Df^n(\bar{x}_0) - Df^n(\bar{x}))v_n(\bar{x})\| + \|Df^n(\bar{x})v_n(\bar{x})\|) \\ &\le \frac{1}{\lambda_n(\bar{x}_0)}\left(\sup_{\bar{y}\in\mathbb{T}^2}\|D^2 f^n(\bar{y})\|\|\bar{x}_0 - \bar{x}\| + \mu_n(\bar{x})\right) \\ &\le \frac{M}{c}\|\bar{x}_0 - \bar{x}\| + \frac{1}{c^2 n^2}.\end{aligned}$$

Hence

$$\|u_n(\bar{x}_0) - u_n(\bar{x})\| \le L\|\bar{x}_0 - \bar{x}\| + \frac{d}{n^2} \tag{4}$$

and

$$|\langle u_n(\bar{x}_0), u_{n+1}(\bar{x}_0)\rangle - \langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle| \le 2L\|\bar{x}_0 - \bar{x}\| + \frac{2d}{n^2},$$

where $L = \sqrt{2}M/c$ and $d = \sqrt{2}/c^2$. Since

$$\langle u_n(\bar{x}_0), u_{n+1}(\bar{x}_0)\rangle = 1 - \frac{1}{2}\|u_n(\bar{x}_0) - u_{n+1}(\bar{x}_0)\|^2 \ge 1 - \frac{2K^2}{n^2},$$

it follows that

$$\langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle \ge 1 - \frac{2(K^2 + d)}{n^2} - 2L\|\bar{x}_0 - \bar{x}\|$$

for any natural n.

Choose $n_0 \in \mathbb{N}$ such that $1 - 2(K^2 + d)/n^2 > 1/2$ for $n \ge n_0$ and fix $r = 1/4L$. Suppose that $\|\bar{x}_0 - \bar{x}\| < r$. Then

$$\langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle \ge \tfrac{1}{2} - 2Lr = 0$$

for $n \geq n_0$. By Lemma 4, $\lim_{n\to\infty} u_n(\bar{x}) = u(\bar{x})$. However, letting $n \to \infty$ in (4), we obtain

$$\|u(\bar{x}_0) - u(\bar{x})\| \leq L\|\bar{x}_0 - \bar{x}\|.$$

Similarly we can prove that if $\|\bar{x}_0 - \bar{x}\| < r$, then

$$\|v(\bar{x}_0) - v(\bar{x})\| \leq L\|\bar{x}_0 - \bar{x}\|. \qquad \square$$

Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^3-diffeomorphism. Assume that f has linear growth of the derivative and the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in the space $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$. Let $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ be sequences of $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$ such that

$$\langle u_n(\bar{x}), u_{n+1}(\bar{x})\rangle \geq 0 \quad \text{and} \quad \langle v_n(\bar{x}), v_{n+1}(\bar{x})\rangle \geq 0$$

for every $\bar{x} \in \mathbb{R}^2$ and $n \in \mathbb{N}$. By Lemma 4, there exist $\mathbb{Z}^2$-periodic functions $u, v : \mathbb{R}^2 \to S^1$ such that

$$\lim_{n\to\infty} u_n(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \lim_{n\to\infty} v_n(\bar{x}) = v(\bar{x})$$

for every $\bar{x} \in \mathbb{R}^2$. By $p : \mathbb{R}^2 \to \mathbb{PR}(1)$ we mean the projection $\mathbb{R}^2$ on the real projection space $\mathbb{PR}(1)$. By Lemma 6, the functions $p \circ u, p \circ v : \mathbb{R}^2 \to \mathbb{PR}(1)$ are Lipschitz continuous. It follows that there exist Lipschitz functions $\tilde{u}, \tilde{v} : \mathbb{R}^2 \to S^1$ such that $p \circ \tilde{u} = p \circ u$ and $p \circ \tilde{v} = p \circ v$. Since $u : \mathbb{R}^2 \to S^1$ is $\mathbb{Z}^2$-periodic,

$$p \circ \tilde{u}(x_1 + 1, x_2) = p \circ u(x_1 + 1, x_2) = p \circ u(x_1, x_2) = p \circ \tilde{u}(x_1, x_2)$$

for every $(x_1, x_2) \in \mathbb{R}^2$. Therefore there exists a function $\varepsilon : \mathbb{R}^2 \to \{-1, 1\}$ such that

$$\tilde{u}(x_1 + 1, x_2) = \varepsilon(x_1, x_2)\tilde{u}(x_1, x_2).$$

Since $\varepsilon(x_1, x_2) = \langle \tilde{u}(x_1, x_2), \tilde{u}(x_1 + 1, x_2)\rangle$, the function ε is continuous, hence ε is constant. It follows that

$$\tilde{u}(x_1 + 2, x_2) = \varepsilon\tilde{u}(x_1 + 1, x_2) = \tilde{u}(x_1, x_2)$$

for any $(x_1, x_2) \in \mathbb{R}^2$. Similarly,

$$\tilde{u}(x_1, x_2 + 2) = \tilde{u}(x_1, x_2) \quad \text{and} \quad \tilde{v}(x_1 + 2, x_2) = \tilde{v}(x_1, x_2 + 2) = \tilde{v}(x_1, x_2)$$

for any $(x_1, x_2) \in \mathbb{R}^2$.

Let $\rho : \mathbb{R}^2(\mathbb{T}^2) \to \mathbb{R}^2(\mathbb{T}^2)$ denote the endomorphism $\rho(x_1, x_2) = (2x_1, 2x_2)$. Then the functions $\widehat{u} = \tilde{u} \circ \rho$ and $\widehat{v} = \tilde{v} \circ \rho$ are $\mathbb{Z}^2$-periodic. From this, we obtain the following conclusion.

COROLLARY 7. *Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^3-diffeomorphism. Assume that f has linear growth of the derivative and the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$. Then there exist Lipschitz functions $\tilde{u}, \tilde{v} : \mathbb{R}^2 \to S^1$ such that the functions $\widehat{u} = \tilde{u} \circ \rho$ and $\widehat{v} = \tilde{v} \circ \rho$ are $\mathbb{Z}^2$-periodic, $\tilde{u}(\bar{x}) \perp \tilde{v}(\bar{x})$ and*

$$\lim_{n\to\infty} \frac{1}{n}(\|Df^n(\bar{x})\tilde{u}(\bar{x})\| - \|Df^n(\bar{x})\|) = \lim_{n\to\infty} \frac{1}{n}\|Df^n(\bar{x})\tilde{v}(\bar{x})\| = 0$$

for every $\bar{x} \in \mathbb{R}^2$.

For a given measure-preserving C^3-diffeomorphism $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ we will denote by $\widehat{f} : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ the measure-preserving C^3-diffeomorphism

$$\widehat{f}(x_1, x_2) = (\tfrac{1}{2} f_1(2x_1, 2x_2), \tfrac{1}{2} f_2(2x_1, 2x_2)). \tag{5}$$

Note that $f \circ \rho = \rho \circ \widehat{f}$. Moreover, $D^k \widehat{f}^n = 2^{k-1} D^k f^n \circ \rho$ for any natural k and

$$\|D\widehat{f}^n(\bar{x})\widehat{u}(\bar{x})\| - \|D\widehat{f}^n(\bar{x})\| = \|Df^n(\rho\bar{x})\tilde{u}(\rho\bar{x})\| - \|Df^n(\rho\bar{x})\|$$

and

$$\|D\widehat{f}^n(\bar{x})\widehat{v}(\bar{x})\| = \|Df^n(\rho\bar{x})\tilde{v}(\rho\bar{x})\|.$$

From this, we obtain the following corollary.

COROLLARY 8. *Suppose that $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is a measure-preserving C^3-diffeomorphism with linear growth of the derivative such that the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$. Then the measure-preserving C^3-diffeomorphism $\widehat{f} : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ satisfies the following conditions:*

- *$\widehat{f}$ has linear growth of the derivative;*
- *the sequence $\{n^{-1}D\widehat{f}^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$;*
- *there exist Lipschitz functions $\widehat{u}, \widehat{v} : \mathbb{T}^2 \to S^1$ such that $\widehat{u}(\bar{x}) \perp \widehat{v}(\bar{x})$ and*

$$\lim_{n\to\infty} \frac{1}{n}(\|D\widehat{f}^n(\bar{x})\widehat{u}(\bar{x})\| - \|D\widehat{f}^n(\bar{x})\|) = \lim_{n\to\infty} \frac{1}{n}\|D\widehat{f}^n(\bar{x})\widehat{v}(\bar{x})\| = 0$$

for every $\bar{x} \in \mathbb{T}^2$.

4. *A few properties of $\widehat{f}$*

In this section we prove a few properties of the diffeomorphism $\widehat{f}$ which we will need in the following sections. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving automorphism of standard Borel space. We will denote by $\mathcal{A}_T$ the σ-algebra of $\mathcal{B}$-measurable T-invariant sets.

LEMMA 9. *If there exists $c > 0$ such that $\mu(A) \geq c$ for every set $A \in \mathcal{A}_T$ with positive measure, then the σ-algebra $\mathcal{A}_T$ is finite.*

Proof. Consider the family $\mathcal{S} = \{A \in \mathcal{A}_T : \mu(A) \geq c\}$ endowed with the order given by the relation of inclusion. Let $\{A_\gamma : \gamma \in \Gamma\}$ be a chain in $\mathcal{S}$. Then $\bigcap_{\gamma\in\Gamma} A_\gamma \in \mathcal{A}_T$. Since $\mu(A_\gamma) \geq c$ for every $\gamma \in \Gamma$, we conclude that $\mu\big(\bigcap_{\gamma\in\Gamma} A_\gamma\big) \geq c > 0$, hence that $\bigcap_{\gamma\in\Gamma} A_\gamma \in \mathcal{S}$. By the Kuratowski–Zorn lemma, for any $A \in \mathcal{S}$ there exists a minimal set $B \in \mathcal{S}$ with $B \subset A$. It follows easily that we can find a finite collection $\{A_1, \ldots, A_k\}$ pairwise disjoint minimal sets in $\mathcal{S}$ such that $\mu\big(\bigcup_{i=1}^k A_k\big) = 1$. Therefore $\mathcal{A}_T$ is generated by the sets $A_1, \ldots, A_k$. □

LEMMA 10. *If $\mathcal{A}_T$ is finite, then $\mathcal{A}_{T^m}$ is finite for any natural m.*

Proof. Let $\{A_1, \ldots, A_k\}$ be a collection of pairwise disjoint sets, which generates the σ-algebra $\mathcal{A}_T$. Suppose that $A \in \mathcal{A}_{T^m}$ and $\mu(A) > 0$. Then $\bigcup_{i=0}^{m-1} T^i A \in \mathcal{A}_T$. Hence

$$\mu(A) \geq \frac{1}{m}\mu\bigg(\bigcup_{i=0}^{m-1} T^i A\bigg) \geq \frac{1}{m} \min_{1\leq j\leq k} \mu(A_j) > 0.$$

Lemma 9 now shows that $\mathcal{A}_{T^m}$ is finite, which completes the proof. □

Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a smooth measure-preserving diffeomorphism. Represent f as

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + \widetilde{f}_1(x_1, x_2), a_{21}x_1 + a_{22}x_2 + \widetilde{f}_2(x_1, x_2)),$$

where $[a_{ij}]_{i,j=1,2} \in GL_2(\mathbb{Z})$ and $\widetilde{f}_1, \widetilde{f}_2 : \mathbb{T}^2 \to \mathbb{R}$. Then

$$\widehat{f}(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + \tfrac{1}{2}\widetilde{f}_1(2x_1, 2x_2), a_{21}x_1 + a_{22}x_2 + \tfrac{1}{2}\widetilde{f}_2(2x_1, 2x_2)).$$

Let $\sigma_1, \sigma_2 : \mathbb{T}^2 \to \mathbb{T}^2$ denote the diffeomorphisms $\sigma_1(x_1, x_2) = (x_1 + \frac{1}{2}, x_2)$, $\sigma_2(x_1, x_2) = (x_1, x_2 + \frac{1}{2})$. Then $\sigma_i \circ \sigma_i = \mathrm{Id}$, $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$ and $\rho \circ \sigma_i = \rho$ for $i = 1, 2$. Let $\varepsilon \in M_2(\mathbb{Z}/2\mathbb{Z})$ be defined by $\varepsilon_{ij} = 2\{a_{ij}/2\}$ for $i, j \in \{1, 2\}$. Then

$$\widehat{f} \circ \sigma_j = \sigma_1^{\varepsilon_{1j}} \circ \sigma_2^{\varepsilon_{2j}} \circ \widehat{f}$$

for $j = 1, 2$. We have $\det \varepsilon \neq 0$, because $\widehat{f}$ is a bijection. It follows that the matrix ε is equal to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore $\varepsilon^6 = \mathrm{Id}$ over the field $\mathbb{Z}/2\mathbb{Z}$. Hence

$$\widehat{f}^6 \circ \sigma_1^{\varepsilon_1} \circ \sigma_2^{\varepsilon_2} = \sigma_1^{\varepsilon_1} \circ \sigma_2^{\varepsilon_2} \circ \widehat{f}^6 \tag{6}$$

for any $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.

LEMMA 11. *If $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is ergodic, then the σ-algebra $\mathcal{A}_{\widehat{f}}$ is finite.*

Proof. Let $A \in \mathcal{A}_{\widehat{f}}$ with $\lambda(A) > 0$. Set $A' = A \cup \sigma_1 A \cup \sigma_2 A \cup \sigma_1\sigma_2 A$. Then $A' \in \mathcal{A}_{\widehat{f}}$ and $\sigma_1 A' = \sigma_2 A' = A'$. Next set $A'' = A' \cap [0, \frac{1}{2}) \times [0, \frac{1}{2})$. Then $A' = A'' \cup \sigma_1 A'' \cup \sigma_2 A'' \cup \sigma_1\sigma_2 A''$ and

$$\begin{aligned} f(\rho A'') &= \rho \circ \widehat{f}(A'') = \rho(\widehat{f}(A'') \cup \sigma_1 \widehat{f}(A'') \cup \sigma_2 \widehat{f}(A'') \cup \sigma_1\sigma_2 \widehat{f}(A'')) \\ &= \rho \circ \widehat{f}(A'' \cup \sigma_1 A'' \cup \sigma_2 A'' \cup \sigma_1\sigma_2 A'') = \rho \circ \widehat{f}(A') = \rho(A') \\ &= \rho(A'' \cup \sigma_1 A'' \cup \sigma_2 A'' \cup \sigma_1\sigma_2 A'') = \rho A''. \end{aligned}$$

By the ergodicity of f, $\lambda(\rho A'') = 1$. Therefore

$$\lambda(A \cup \sigma_1 A \cup \sigma_2 A \cup \sigma_1\sigma_2 A) = \lambda(A') = \lambda(\rho^{-1}(\rho A'')) = \lambda(\rho A'') = 1.$$

It follows that $\lambda(A) \geq 1/4$. Now we can apply Lemma 9 and the proof is complete. □

LEMMA 12. *If $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ is ergodic, then there exists a dense subset $A \subset \mathbb{T}^2$ and an increasing sequence $\{n_i\}_{i \in \mathbb{N}}$ of natural numbers such that $\widehat{f}^{n_i} \bar{x} \to \bar{x}$ for every $\bar{x} \in A$.*

Proof. By Lemmas 10 and 11, the σ-algebra $\mathcal{A}_{\widehat{f}^6}$ is finite. Let $\{A_1, \ldots, A_s\}$ be a collection of measurable pairwise disjoint sets (with positive measure), which generates $\mathcal{A}_{\widehat{f}^6}$. Let $\mathcal{U}_1 = \{U \in \mathcal{U} : \lambda(U \cap A_1) = 0\}$, where $\mathcal{U}$ is the family all open subsets of $\mathbb{T}^2$. Set

$$B_1 = A_1 \Big\backslash \bigcup_{U \in \mathcal{U}_1} U.$$

Then $\lambda(B_1) = \lambda(A_1)$ and $\widehat{f}^6 B_1 = B_1 \bmod \lambda$. Next set $B = \bigcap_{n\in\mathbb{Z}} \widehat{f}^{6n} B_1$. Then $\widehat{f}^6 B = B$ and $B \cap U \neq \emptyset$, $U \in \mathcal{U}$ implies $\lambda(B \cap U) > 0$. Now consider the measure-preserving homeomorphism $\widehat{f}^6 : (B, \lambda|B) \to (B, \lambda|B)$ (with the induced topology). Since $\widehat{f}^6 : (B, \lambda|B) \to (B, \lambda|B)$ and $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ are ergodic, there exists $\bar{x}_0 \in B$ such that the sequence $\{\widehat{f}^{6n}\bar{x}_0\}_{n\in\mathbb{N}}$ is dense in B and the sequence $\{f^n \rho \bar{x}_0\}_{n\in\mathbb{N}}$ is dense in $\mathbb{T}^2$. Choose an increasing sequence $\{m_i\}_{i\in\mathbb{N}}$ such that $\widehat{f}^{6m_i}\bar{x}_0 \to \bar{x}_0$. Define

$$A = \{\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\widehat{f}^n(\bar{x}_0) : \varepsilon_1, \varepsilon_2 \in \{-1, 1\},\ n \in \mathbb{N}\}$$

and $n_i = 6m_i$ for any $i \in \mathbb{N}$. First note that $\widehat{f}^{n_i}\bar{x} \to \bar{x}$ for every $\bar{x} \in A$. Indeed, let $\bar{x} = \sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\widehat{f}^n(\bar{x}_0)$, where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ and $n \in \mathbb{N}$. From (6), it follows that

$$\widehat{f}^{n_i}\bar{x} = \widehat{f}^{6m_i}\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\widehat{f}^n\bar{x}_0 = \sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\widehat{f}^n\widehat{f}^{6m_i}\bar{x}_0 \to \sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\widehat{f}^n\bar{x}_0 = \bar{x}.$$

Next we show that A is dense in $\mathbb{T}^2$. First observe that $\rho A = \{f^n \rho \bar{x}_0 : n \in \mathbb{N}\}$ is dense in $\mathbb{T}^2$ and $\sigma_1 A = \sigma_2 A = A$. Let $\bar{y} \in \mathbb{T}^2$. By the density of ρA, there exists a sequence $\{\bar{y}_n\}_{n\in\mathbb{N}}$ in A such that $\rho\bar{y}_n \to \rho\bar{y}$. Let $\{\bar{y}_{s_i}\}_{i\in\mathbb{N}}$ be a convergent subsequence with $\bar{y}' = \lim_{i\to\infty}\bar{y}_{s_i}$. Then

$$\rho\bar{y} = \lim_{i\to\infty}\rho\bar{y}_{s_i} = \rho\bar{y}'.$$

Therefore there exist $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$ such that $\bar{y} = \sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\bar{y}'$. Hence

$$\bar{y} = \lim_{i\to\infty}\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\bar{y}_{s_i} \quad \text{and} \quad \sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}\bar{y}_{s_i} \in \sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2}A = A,$$

which completes the proof. □

5. *Proof of the Main theorem*

We will use the symbol h_{x_i} to denote the partial derivative $\partial h/\partial x_i$. Let $\varepsilon \in \{-1, 1\}$ be a number such that $\varepsilon = \det Df(\bar{x})$ for any $\bar{x} \in \mathbb{T}^2$. In this section we will show the following result, which leads to the Main theorem.

THEOREM 13. *Let $f : (\mathbb{T}^2, \lambda) \to (\mathbb{T}^2, \lambda)$ be a measure-preserving C^3-diffeomorphism. Assume that:*

- *the σ-algebra $\mathcal{A}_f$ is finite;*
- *there exists a dense subset $A \subset \mathbb{T}^2$ and an increasing sequence $\{s_i\}_{i\in\mathbb{N}}$ of natural numbers such that $f^{s_i}\bar{x} \to \bar{x}$ for every $\bar{x} \in A$;*
- *f has linear growth of the derivative;*
- *the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$; and*
- *there exist Lipschitz functions $u, v : \mathbb{T}^2 \to S^1$ such that $u(\bar{x}) \perp v(\bar{x})$ and*

$$\lim_{n\to\infty}\frac{1}{n}(\|Df^n(\bar{x})u(\bar{x})\| - \|Df^n(\bar{x})\|) = \lim_{n\to\infty}\frac{1}{n}\|Df^n(\bar{x})v(\bar{x})\| = 0$$

for every $\bar{x} \in \mathbb{T}^2$.

Then f is algebraically conjugate to a skew product of an irrational rotation on the circle and a circle C^3-cocycle with non-zero degree.

This immediately gives the following.

Proof of Theorem 3. By Corollary 8, Lemmas 11 and 12 and Theorem 13, there exists a group automorphism $B : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$B \circ \widehat{f} \circ B^{-1}(x_1, x_2) = T_{\alpha,\varphi}(x_1, x_2) = (x_1 + \alpha, x_2 + \varphi(x_1)),$$

where α is irrational and $\varphi : \mathbb{T} \to \mathbb{T}$ is a C^3-cocycle with non-zero degree. Since ρ commutes with B and $B^{-1} \circ \sigma_1 = \sigma_1^{\varepsilon_1} \circ \sigma_2^{\varepsilon_2} \circ B^{-1}$, where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, we have

$$\begin{aligned}(2x_1 + 2\alpha, 2x_2 + 2\varphi(x_1 + \tfrac{1}{2})) &= \rho \circ T_{\alpha,\varphi} \circ \sigma_1(x_1, x_2)\\ &= B \circ f \circ \rho \circ \sigma_1^{\varepsilon_1} \circ \sigma_2^{\varepsilon_2} \circ B^{-1}(x_1, x_2)\\ &= B \circ f \circ \rho \circ B^{-1}(x_1, x_2) = \rho \circ T_{\alpha,\varphi}(x_1, x_2)\\ &= (2x_1 + 2\alpha, 2x_2 + 2\varphi(x_1)).\end{aligned}$$

It follows that $\varphi(x + 1/2) = \varphi(x) + d(\varphi)/2$. Define

$$\tilde{\varphi}(x) = 2\varphi(\tfrac{1}{2}x).$$

Then $\tilde{\varphi} : \mathbb{T} \to \mathbb{T}$ and $d(\tilde{\varphi}) = d(\varphi) \neq 0$. Moreover,

$$\widehat{B \circ f \circ B^{-1}} = B \circ \widehat{f} \circ B^{-1} = T_{\alpha,\varphi} = \widehat{T_{2\alpha,\tilde{\varphi}}}.$$

Therefore $B \circ f \circ B^{-1} = T_{2\alpha,\tilde{\varphi}}$. □

Remark. In the remainder of the paper assume that the system $[u(\bar{x}), v(\bar{x})]$ has a positive orientation, i.e. there exist Lipschitz functions $a, b : \mathbb{T}^2 \to \mathbb{R}$ such that

$$u(\bar{x}) = \begin{bmatrix} a(\bar{x}) \\ b(\bar{x}) \end{bmatrix}, \quad v(\bar{x}) = \begin{bmatrix} -b(\bar{x}) \\ a(\bar{x}) \end{bmatrix}$$

and $a^2(\bar{x}) + b^2(\bar{x}) = 1$ for any $\bar{x} \in \mathbb{T}^2$.

To prove Theorem 13, we need the following lemmas.

LEMMA 14. *Under the assumptions of Theorem 13, $u, v \in C^1(\mathbb{T}^2, \mathbb{R}^2)$.*

Proof. Since the sequence $\{n^{-1}Df^n\}_{n\in\mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$, there exists a subsequence $\{n_i^{-1}Df^{n_i}\}_{i\in\mathbb{N}}$ convergent to a function h in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$. Then $0 < c \leq \|h(\bar{x})\| \leq C$ for any $\bar{x} \in \mathbb{T}^2$. Since $n^{-1}Df^n(\bar{x})v(\bar{x}) \to 0$, we have $h(\bar{x})v(\bar{x}) = 0$ for any $\bar{x} \in \mathbb{T}^2$. Therefore

$$h(\bar{x}) = \begin{bmatrix} c(\bar{x})a(\bar{x}) & c(\bar{x})b(\bar{x}) \\ d(\bar{x})a(\bar{x}) & d(\bar{x})b(\bar{x}) \end{bmatrix},$$

where $c, d : \mathbb{T}^2 \to \mathbb{R}$ are Lipschitz functions given by

$$\begin{bmatrix} c(\bar{x}) \\ d(\bar{x}) \end{bmatrix} = h(\bar{x}) \begin{bmatrix} a(\bar{x}) \\ b(\bar{x}) \end{bmatrix}.$$

Then $\|h(\bar{x})\|^2 = c^2(\bar{x}) + d^2(\bar{x})$. Let $g : \mathbb{T}^2 \to \mathbb{R}$ be given by $g(\bar{x}) = \|h(\bar{x})\|$. Since $g(\bar{x}) = \|h(\bar{x})\| = \sqrt{\sum_{i,j\in\{1,2\}}(h^{ij}(\bar{x}))^2}$ ($h = [h^{ij}]_{i,j\in\{1,2\}}$) and $0 < c \leq g(\bar{x}) \leq C$, g is a C^1-function.

Fix $\tilde{x}_2 \in \mathbb{R}$. Since $a, c, d : \mathbb{T}^2 \to \mathbb{R}$ are Lipschitz, for almost all $x_1 \in \mathbb{R}$ there exist partial derivatives $a_{x_1}(x_1, \tilde{x}_2)$, $c_{x_1}(x_1, \tilde{x}_2)$, $d_{x_1}(x_1, \tilde{x}_2)$ and $a_{x_1}(\,\cdot\,, \tilde{x}_2) \in L^1(\mathbb{T})$. Moreover, $a(x_1, \tilde{x}_2) - a(0, \tilde{x}_2) = \int_0^{x_1} a_{x_1}(x, \tilde{x}_2)\, dx$ for any $x_1 \in \mathbb{R}$. Then

$$h^{11}_{x_1}(x_1, \tilde{x}_2) = \frac{\partial}{\partial x_1}(c \cdot a)(x_1, \tilde{x}_2) = (c_{x_1} \cdot a + c \cdot a_{x_1})(x_1, \tilde{x}_2)$$

and

$$h^{21}_{x_1}(x_1, \tilde{x}_2) = \frac{\partial}{\partial x_1}(d \cdot a)(x_1, \tilde{x}_2) = (d_{x_1} \cdot a + d \cdot a_{x_1})(x_1, \tilde{x}_2)$$

for almost all $x_1 \in \mathbb{R}$. Hence

$$\begin{aligned}(h^{11}_{x_1} \cdot c + h^{21}_{x_1} \cdot d)(x_1, \tilde{x}_2) &= ((c_{x_1} \cdot c + d_{x_1} \cdot d) \cdot a)(x_1, \tilde{x}_2) + ((c^2 + d^2) \cdot a_{x_1})(x_1, \tilde{x}_2)\\ &= ((g^2)_{x_1} \cdot a)(x_1, \tilde{x}_2) + (g^2 \cdot a_{x_1})(x_1, \tilde{x}_2)\end{aligned}$$

and

$$a_{x_1}(x_1, \tilde{x}_2) = \frac{h^{11}_{x_1} \cdot c + h^{21}_{x_1} \cdot d - (g^2)_{x_1} \cdot a}{g^2}(x_1, \tilde{x}_2)$$

for almost all $x_1 \in \mathbb{R}$. It follows that there exists a continuous function $e : \mathbb{T}^2 \to \mathbb{R}$ such that $a_{x_1}(x_1, \tilde{x}_2) = e(x_1, \tilde{x}_2)$ for almost all $x_1 \in \mathbb{R}$. Therefore

$$a(x_1, \tilde{x}_2) - a(0, \tilde{x}_2) = \int_0^{x_1} a_{x_1}(x, \tilde{x}_2)\, dx = \int_0^{x_1} e(x, \tilde{x}_2)\, dx$$

for any $x_1 \in \mathbb{R}$. Hence for every $x_1 \in \mathbb{R}$ there exists $a_{x_1}(x_1, \tilde{x}_2)$ and is equal to $e(x_1, \tilde{x}_2)$. Therefore, for every $\bar{x} \in \mathbb{R}$ there exists $a_{x_1}(\bar{x})$ and $a_{x_1} : \mathbb{T}^2 \to \mathbb{R}$ is continuous. Similarly, we can prove that $a_{x_2}, b_{x_1}, b_{x_2} : \mathbb{T}^2 \to \mathbb{R}$ are continuous. It follows that $a, b \in C^1(\mathbb{T}^2, \mathbb{R})$ and finally that $u, v \in C^1(\mathbb{T}^2, \mathbb{R}^2)$. □

Let $\langle B, \|\cdot\|\rangle$ be a Banach space and let $r \in \mathbb{N} \cup \{0\}$. Consider the space

$$C^{r+L}(\mathbb{T}^2, B) = \left\{ f \in C^r(\mathbb{T}^2, B) : \sup_{\substack{\bar{x}, \bar{y} \in \mathbb{T}^2,\\ \bar{x} \neq \bar{y}}} \frac{\|D^r f(\bar{x}) - D^r f(\bar{y})\|}{\|\bar{x} - \bar{y}\|} < \infty \right\}$$

endowed with the norm given by

$$\|f\|_{r+L} = \max\left\{ \|f\|_r, \sup_{\substack{\bar{x}, \bar{y} \in \mathbb{T}^2,\\ \bar{x} \neq \bar{y}}} \frac{\|D^r f(\bar{x}) - D^r f(\bar{y})\|}{\|\bar{x} - \bar{y}\|} \right\}.$$

Applying the Ascoli theorem, we get immediately the following lemma.

LEMMA 15. *Let r be a natural number and let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $C^{r+L}(\mathbb{T}^2, B)$. Then there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that $f_{n_i} \to f$ in $C^r(\mathbb{T}^2, B)$. Moreover, $f \in C^{r+L}(\mathbb{T}^2, B)$ and $\|f\|_{r+L} \leq \limsup_{n\to\infty} \|f_n\|_{r+L}$.*

This gives the following result by the diagonal process.

LEMMA 16. *Under the assumptions of Theorem 13, there exists an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that $f^{n_i}\bar{x} \to \bar{x}$ for every $\bar{x} \in A$ and*

$$\lim_{i\to\infty} \frac{1}{n_i} Df^{n_i+k} = h_k$$

in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$ for any integer k. Moreover, $h_k \in C^{1+L}(\mathbb{T}^2, M_2(\mathbb{R}))$ and

$$\|h_k\|_{1+L} \le \limsup_{n\to\infty} \frac{1}{n}\|Df^n\|_{1+L} \le \limsup_{n\to\infty} \frac{1}{n}\|Df^n\|_2$$

for any integer k.

Let $\{n_i\}_{i\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $f^{n_i}\bar{x} \to \bar{x}$ for every $\bar{x} \in A$ and

$$\lim_{i\to\infty} \frac{1}{n_i} Df^{n_i+k} = h_k$$

in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$ for any natural k. It follows that the sequence $\{h_k\}_{k\in\mathbb{Z}}$ is bounded in $C^{1+L}(\mathbb{T}^2, M_2(\mathbb{R}))$ and $0 < c \le \|h_k(\bar{x})\| \le C$ for any $\bar{x} \in \mathbb{T}^2$ and $k \in \mathbb{Z}$. Since $n^{-1}Df^n(\bar{x})v(\bar{x}) \to 0$, we have $h_k(\bar{x})v(\bar{x}) = 0$. Then $\|h_k(\bar{x})\| = \sqrt{\sum_{i,j\in\{1,2\}}(h_k^{ij}(\bar{x}))^2}$ ($h_k = [h_k^{ij}]_{i,j\in\{1,2\}}$), by $\det h_k(\bar{x}) = 0$. For every integer k, let $g_k : \mathbb{T}^2 \to \mathbb{R}$ be given by $g_k(\bar{x}) = \|h_k(\bar{x})\|$. Then $g_k \in C^{1+L}(\mathbb{T}^2, \mathbb{R})$. Moreover, the sequence $\{g_k\}_{k\in\mathbb{Z}}$ is bounded in $C^{1+L}(\mathbb{T}^2, \mathbb{R})$.

LEMMA 17. *For every integer k there exist $s_k : \mathbb{T}^2 \to \mathbb{R}$ and $\varepsilon_k \in \{-1, 1\}$ such that*

$$[u(f^k\bar{x})\ v(f^k\bar{x})]\begin{bmatrix} \varepsilon_k g_k(\bar{x})/g_0(f^k\bar{x}) & 0 \\ s_k(\bar{x}) & \varepsilon^k\varepsilon_k g_0(f^k\bar{x})/g_k(\bar{x}) \end{bmatrix} = Df^k(\bar{x})[u(\bar{x})\ v(\bar{x})]$$

for any $\bar{x} \in \mathbb{T}^2$.

Proof. Fix $k \in \mathbb{Z}$. Since

$$\lim_{n\to\infty} \frac{1}{n}(\|Df^n(\bar{x})u(\bar{x})\| - \|Df^n(\bar{x})\|) = \lim_{n\to\infty} \frac{1}{n}\|Df^n(\bar{x})v(\bar{x})\| = 0,$$

we have

$$\lim_{i\to\infty} \frac{1}{n_i}\|Df^{n_i}(f^k\bar{x})u(f^k\bar{x})\| = g_0(f^k\bar{x}), \quad \lim_{i\to\infty} \frac{1}{n_i}\|Df^{n_i}(f^k\bar{x})v(f^k\bar{x})\| = 0.$$

Let $e_1, e_2 : \mathbb{T}^2 \to \mathbb{R}$ be functions such that

$$\|(g_k(\bar{x})/g_0(f^k\bar{x}))(Df^k(\bar{x}))^{-1}u(f^k\bar{x}) + e_1(\bar{x})v(\bar{x})\| = 1,$$
$$\|e_2(\bar{x})(Df^k(\bar{x}))^{-1}v(f^k\bar{x})\| = 1.$$

Set

$$u'(\bar{x}) = (g_k(\bar{x})/g_0(f^k\bar{x}))(Df^k(\bar{x}))^{-1}u(f^k\bar{x}) + e_1(\bar{x})v(\bar{x}),$$
$$v'(\bar{x}) = e_2(\bar{x})(Df^k(\bar{x}))^{-1}v(f^k\bar{x}).$$

Then

$$\lim_{i\to\infty}\frac{1}{n_i+k}(\|Df^{n_i+k}(\bar{x})u'(\bar{x})\|-\|Df^{n_i+k}(\bar{x})\|)$$
$$=\lim_{i\to\infty}\left\|(g_k(\bar{x})/g_0(f^k\bar{x}))\frac{1}{n_i}Df^{n_i}(f^k\bar{x})u(f^k\bar{x})+e_1(\bar{x})\frac{1}{n_i}Df^{n_i+k}(\bar{x})v(\bar{x})\right\|-g_k(\bar{x})$$
$$=(g_k(\bar{x})/g_0(f^k\bar{x}))\|h_0(f^k\bar{x})\|-g_k(\bar{x})=0$$

and

$$\lim_{i\to\infty}\frac{1}{n_i+k}\|Df^{n_i+k}(\bar{x})v'(\bar{x})\|=\lim_{i\to\infty}\|e_2(\bar{x})\frac{1}{n_i}Df^{n_i}(f^k\bar{x})v(f^k\bar{x})\|=0.$$

By Lemma 5, there exist functions $\varepsilon_{1k},\varepsilon_{2k}:\mathbb{T}^2\to\{-1,1\}$ such that $u'=\varepsilon_{1k}u$ and $v'=\varepsilon_{2k}v$. Therefore

$$Df^k(\bar{x})^{-1}[u(f^k\bar{x})\ v(f^k\bar{x})]=[u(\bar{x})\ v(\bar{x})]\begin{bmatrix}\varepsilon_{1k}(\bar{x})g_0(f^k\bar{x})/g_k(\bar{x}) & 0\\ -e_1(\bar{x})g_0(f^k\bar{x})/g_k(\bar{x}) & \varepsilon_{2k}(\bar{x})/e_2(\bar{x})\end{bmatrix}.$$

It follows that ε_{1k} is continuous, hence that it is constant. Let $\varepsilon_k\in\{-1,1\}$ be a number such that $\varepsilon_k=\varepsilon_{1k}(\bar{x})$ for any $\bar{x}\in\mathbb{T}^2$. Since $\det[u(\bar{x})v(\bar{x})]=1$ and $\det Df^k(\bar{x})=\varepsilon^k$ for any $\bar{x}\in\mathbb{T}^2$, we have

$$\varepsilon_{2k}(\bar{x})/e_2(\bar{x})=\varepsilon^k\varepsilon_k\,g_k(\bar{x})/g_0(f^k\bar{x}).$$

Set $s_k(\bar{x})=\varepsilon^k\,e_1(\bar{x})g_0(f^k\bar{x})/g_k(\bar{x})$. Then

$$Df^k(\bar{x})^{-1}[u(f^k\bar{x})v(f^k\bar{x})]=[u(\bar{x})v(\bar{x})]\begin{bmatrix}\varepsilon_k\,g_0(f^k\bar{x})/g_k(\bar{x}) & 0\\ -\varepsilon^k s_k(\bar{x}) & \varepsilon^k\varepsilon_k\,g_k(\bar{x})/g_0(f^k\bar{x})\end{bmatrix}$$

and finally

$$[u(f^k\bar{x})v(f^k\bar{x})]\begin{bmatrix}\varepsilon_k\,g_k(\bar{x})/g_0(f^k\bar{x}) & 0\\ s_k(\bar{x}) & \varepsilon^k\varepsilon_k\,g_0(f^k\bar{x})/g_k(\bar{x})\end{bmatrix}=Df^k(\bar{x})[u(\bar{x})v(\bar{x})].\quad\square$$

LEMMA 18. *There exists a function $g:\mathbb{T}^2\to\mathbb{R}_+$ of class C^{1+L} such that*

$$g_k(\bar{x})/g_0(f^k\bar{x})=g(\bar{x})/g(f^k\bar{x})$$

and $\varepsilon_k=\varepsilon_1^k$ for any integer k.

Proof. For all $k,n\in\mathbb{Z}$ we have

$$\begin{aligned}Df^{n+k}[uv]&=Df^n\circ f^k\cdot Df^k[uv]\\&=Df^n\circ f^k[u\circ f^k\ v\circ f^k]\begin{bmatrix}\varepsilon_k\dfrac{g_k}{g_0\circ f^k} & 0\\ s_k & \varepsilon^k\varepsilon_k\dfrac{g_0\circ f^k}{g_k}\end{bmatrix}\\&=[u\circ f^{n+k}\ v\circ f^{n+k}]\begin{bmatrix}\varepsilon_n\dfrac{g_n\circ f^k}{g_0\circ f^{n+k}} & 0\\ s_n\circ f^k & \varepsilon^n\varepsilon_n\dfrac{g_0\circ f^{n+k}}{g_n\circ f^k}\end{bmatrix}\\&\quad\times\begin{bmatrix}\varepsilon_k\dfrac{g_k}{g_0\circ f^k} & 0\\ s_k & \varepsilon^k\varepsilon_k\dfrac{g_0\circ f^k}{g_k}\end{bmatrix}.\end{aligned}$$

However,

$$Df^{n+k}[uv] = [u \circ f^{n+k}\ v \circ f^{n+k}] \begin{bmatrix} \varepsilon_{n+k} \dfrac{g_{n+k}}{g_0 \circ f^{n+k}} & 0 \\ s_{n+k} & \varepsilon^{n+k}\varepsilon_{n+k} \dfrac{g_0 \circ f^{n+k}}{g_{n+k}} \end{bmatrix}.$$

It follows that $\varepsilon_{n+k} = \varepsilon_n \varepsilon_k$ and

$$\frac{g_{n+k}}{g_0 \circ f^{n+k}} = \frac{g_n \circ f^k}{g_0 \circ f^{n+k}} \frac{g_k}{g_0 \circ f^k}.$$

Hence $\varepsilon_k = \varepsilon_1^k$ and $g_k / g_0 \circ f^k = g_{n+k}/g_n \circ f^k$. Let $\zeta : \mathbb{T}^2 \to \mathbb{R}_+$ be given by

$$\zeta(\bar{x}) = g_1(\bar{x})/g_0(f\bar{x}) = g_{n+1}(\bar{x})/g_n(f\bar{x}).$$

Then $\zeta \in C^{1+L}(\mathbb{T}^2, \mathbb{R})$, $\zeta(\bar{x}) \geq c/C > 0$ for any $\bar{x} \in \mathbb{T}^2$ and

$$g_n = \zeta \cdot \zeta \circ f \cdot \ldots \cdot \zeta \circ f^{n-1} \cdot g_0 \circ f^n$$

for any natural n. Define $\tilde{\zeta} = \log \zeta$ and $\tilde{g}_n = \log g_n$ for any integer n. Then $\tilde{\zeta}, \tilde{g}_n \in C^{1+L}(\mathbb{T}^2, \mathbb{R})$. Since the sequence $\{g_n\}_{n \in \mathbb{Z}}$ is bounded in $C^{1+L}(\mathbb{T}^2, \mathbb{R})$ and $0 < c \leq g_n(\bar{x})$ for any $n \in \mathbb{Z}$ and $\bar{x} \in \mathbb{T}^2$, the sequence $\{\tilde{g}_n\}_{n \in \mathbb{Z}}$ is bounded in $C^{1+L}(\mathbb{T}^2, \mathbb{R})$, too. Moreover,

$$\tilde{g}_n = \sum_{k=0}^{n-1} \tilde{\zeta} \circ f^k + \tilde{g}_0 \circ f^n$$

for any natural n. Set $\tilde{\zeta}^{(n)} = \sum_{k=0}^{n-1} \tilde{\zeta} \circ f^k$ for any natural n. Then

$$\begin{aligned} \tilde{\zeta} - \frac{1}{n}\tilde{\zeta}^{(n)} &= \frac{1}{n}\sum_{k=0}^{n-1}(\tilde{\zeta} - \tilde{\zeta} \circ f^k) = \frac{1}{n}\sum_{k=0}^{n-1}(\tilde{\zeta}^{(k)} - \tilde{\zeta}^{(k)} \circ f) = \frac{1}{n}\sum_{k=0}^{n-1}\tilde{\zeta}^{(k)} - \frac{1}{n}\sum_{k=0}^{n-1}\tilde{\zeta}^{(k)} \circ f \\ &= \frac{1}{n}\sum_{k=0}^{n-1}(\tilde{g}_k - \tilde{g}_0 \circ f^k) - \frac{1}{n}\sum_{k=0}^{n-1}(\tilde{g}_k \circ f - \tilde{g}_0 \circ f^{k+1}) \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\tilde{g}_k - \frac{1}{n}\sum_{k=0}^{n-1}\tilde{g}_k \circ f + \frac{\tilde{g}_0 \circ f^n - \tilde{g}_0}{n}. \end{aligned}$$

Since the sequence $\{n^{-1}\sum_{k=0}^{n-1}\tilde{g}_k\}_{n \in \mathbb{N}}$ is bounded in $C^{1+L}(\mathbb{T}^2, \mathbb{R})$, there exists a subsequence $\{m_i^{-1}\sum_{k=0}^{m_i-1}\tilde{g}_k\}_{i \in \mathbb{N}}$ convergent in $C^1(\mathbb{T}^2, \mathbb{R})$ to a function $\tilde{g} \in C^{1+L}(\mathbb{T}^2, \mathbb{R})$, by Lemma 15. By the ergodic theorem, the sequence $\{n^{-1}\tilde{\zeta}^{(n)}\}_{n \in \mathbb{N}}$ converges a.e. to a $\mathcal{A}_f$-measurable function $\tilde{\zeta}_0 : \mathbb{T}^2 \to \mathbb{R}$. It follows that

$$\tilde{\zeta}(\bar{x}) - \tilde{\zeta}_0(\bar{x}) = \tilde{g}(\bar{x}) - \tilde{g}(f\bar{x})$$

for a.e. $\bar{x} \in \mathbb{T}^2$. Since the function $\tilde{\zeta} - \tilde{g} + \tilde{g} \circ f$ is $\mathcal{A}_f$-measurable and continuous and the σ-algebra $\mathcal{A}_f$ is finite, we conclude that $\tilde{\zeta} - \tilde{g} + \tilde{g} \circ f$ is constant. Hence

$$\tilde{\zeta} - \tilde{g} + \tilde{g} \circ f = \int_{\mathbb{T}^2} \tilde{\zeta}_0 \, d\lambda = \int_{\mathbb{T}^2} \tilde{\zeta} \, d\lambda.$$

Since $\tilde{g}_n = \tilde{\zeta}^{(n)} + \tilde{g}_0 \circ f^n$, we have

$$\int_{\mathbb{T}^2} \tilde{g}_n \, d\lambda = n \int_{\mathbb{T}^2} \tilde{\zeta} \, d\lambda + \int_{\mathbb{T}^2} \tilde{g}_0 \, d\lambda.$$

As the sequence $\{\int_{\mathbb{T}^2} \tilde{g}_n \, d\lambda\}_{n \in \mathbb{N}}$ is bounded, we obtain $\int_{\mathbb{T}^2} \tilde{\zeta} \, d\lambda = 0$. This gives $\tilde{\zeta} = \tilde{g} - \tilde{g} \circ f$ and

$$\tilde{g}_k - \tilde{g}_0 \circ f^k = \tilde{\zeta}^{(k)} = \tilde{g} - \tilde{g} \circ f^k$$

for any natural k. Define $g = \exp \tilde{g}$. Then

$$g_k / g_0 \circ f^k = g / g \circ f^k. \qquad \square$$

Lemmas 17 and 18 now show that

$$[u \circ f^k \;\; v \circ f^k] \begin{bmatrix} \varepsilon_1^k \; g/g \circ f^k & 0 \\ s_k & (\varepsilon\varepsilon_1)^k \; g \circ f^k / g \end{bmatrix} = Df^k [uv].$$

Therefore

$$(g \cdot v) \circ f^k = (\varepsilon_1 \varepsilon)^k Df^k (g \cdot v) \tag{7}$$

and

$$\begin{bmatrix} \varepsilon_1^k \; g/g \circ f^k & 0 \\ s_k & (\varepsilon\varepsilon_1)^k \; g \circ f^k / g \end{bmatrix} \begin{bmatrix} u^T \\ v^T \end{bmatrix} = \begin{bmatrix} u^T \circ f^k \\ v^T \circ f^k \end{bmatrix} Df^k.$$

Hence

$$(g \cdot u^T) \circ f^k \, Df^k = \varepsilon_1^k (g \cdot u^T). \tag{8}$$

LEMMA 19. *For every integer k there exists $\delta_k \in \{-1, 1\}$ such that*

$$\begin{aligned} h_k &= \delta_k \, g_k \begin{bmatrix} -a \cdot b \circ f^k & -b \cdot b \circ f^k \\ a \cdot a \circ f^k & b \cdot a \circ f^k \end{bmatrix} \\ &= \delta_k \, g_k \cdot v \circ f^k \cdot u^T. \end{aligned}$$

Proof. Recall that $\{n_i\}_{i \in \mathbb{N}}$ is the sequence for which $f^{n_i} \bar{x} \to \bar{x}$ for every $\bar{x} \in A$ and

$$\lim_{i \to \infty} \frac{1}{n_i} Df^{n_i + k} = h_k$$

in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$. Therefore $h_k \cdot v = 0$. Let $c_k, d_k \in C^1(\mathbb{T}^2, \mathbb{R})$ be given by

$$\begin{bmatrix} c_k \\ d_k \end{bmatrix} = \frac{1}{g_k} h_k \cdot u.$$

Then $c_k^2 + d_k^2 = 1$ and

$$h_k = g_k \begin{bmatrix} c_k \\ d_k \end{bmatrix} u^T. \tag{9}$$

Suppose that $\bar{x} \in A$. From (8),

$$\frac{1}{n_i} u^T(f^{n_i+k}\bar{x}) Df^{n_i+k}(\bar{x}) = \frac{\varepsilon_1^{n_i+k}}{n_i}(g(\bar{x})/g(f^{n_i+k}\bar{x}))\, u^T(\bar{x}) \to 0.$$

Since $f^{n_i+k}\bar{x} \to f^k\bar{x}$, we have $u^T(f^k\bar{x}) \cdot h_k(\bar{x}) = 0$. From (9), it follows that

$$u^T(f^k\bar{x}) \begin{bmatrix} c_k(\bar{x}) \\ d_k(\bar{x}) \end{bmatrix} u^T(\bar{x}) = 0,$$

hence that

$$u^T(f^k\bar{x}) \begin{bmatrix} c_k(\bar{x}) \\ d_k(\bar{x}) \end{bmatrix} = 0 \tag{10}$$

for any $\bar{x} \in A$. Since the set A is dense in $\mathbb{T}^2$, we see that the equality (10) holds for any $\bar{x} \in \mathbb{T}^2$. It follows that there exists a function $\delta_k : \mathbb{T}^2 \to \{-1, 1\}$ such that

$$\begin{bmatrix} c_k(\bar{x}) \\ d_k(\bar{x}) \end{bmatrix} = \delta_k(\bar{x})\, v(f^k\bar{x}).$$

Since

$$\delta_k(\bar{x}) = v^T(f^k\bar{x}) \begin{bmatrix} c_k(\bar{x}) \\ d_k(\bar{x}) \end{bmatrix},$$

we conclude that δ_k is continuous, hence that it is constant. Therefore

$$h_k = \delta_k\, g_k \cdot v \circ f^k \cdot u^T,$$

by (9). □

LEMMA 20. *$a_{x_1} + b_{x_2} = 0$ and $g_{x_1} b = g_{x_2} a$.*

Proof. Since $n_i^{-1} Df^{n_i} \to h_0$ in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$, it follows that $n_i^{-1} D^2 f^{n_i} \to Dh_0$ uniformly. Let $\tilde{v} = g \cdot v$. Then

$$D\left(\frac{1}{n_i} Df^{n_i} \cdot \tilde{v}\right) = \frac{1}{n_i} D^2 f^{n_i} \cdot \tilde{v} + \frac{1}{n_i} Df^{n_i} \cdot D\tilde{v} \to Dh_0 \cdot \tilde{v} + h_0 \cdot D\tilde{v} = D(h_0 \cdot \tilde{v}) = 0 \tag{11}$$

uniformly, because $h_0 \cdot \tilde{v} = g \cdot h_0 \cdot v = 0$. However, $\tilde{v} \circ f^k = (\varepsilon_1\varepsilon)^k Df^k \cdot \tilde{v}$ (see (7)) implies $(\varepsilon_1\varepsilon)^k D(Df^k\tilde{v}) = D\tilde{v} \circ f^k \cdot Df^k$. Let $\bar{x} \in A$. Then

$$(\varepsilon_1\varepsilon)^{n_i} D\left(\frac{1}{n_i} Df^{n_i}(\bar{x})\tilde{v}(\bar{x})\right) = D\tilde{v}(f^{n_i}\bar{x}) \frac{1}{n_i} Df^{n_i}(\bar{x}) \to D\tilde{v}(\bar{x}) h_0(\bar{x}).$$

From (11), we obtain $D\tilde{v}(\bar{x}) h_0(\bar{x}) = 0$ for any $\bar{x} \in A$. Since A is dense in $\mathbb{T}^2$, it follows that $D\tilde{v} \cdot h_0 = 0$. Hence

$$\begin{bmatrix} -(g \cdot b)_{x_1} & -(g \cdot b)_{x_2} \\ (g \cdot a)_{x_1} & (g \cdot a)_{x_2} \end{bmatrix} \begin{bmatrix} -b \\ a \end{bmatrix} = (\delta_0/g_0) D\tilde{v} \cdot h_0 \cdot u = 0.$$

Since $a_{x_1}a + b_{x_1}b = 0$ and $a_{x_2}a + b_{x_2}b = 0$, we conclude that

$$\begin{aligned} 0 &= (g \cdot b)_{x_1} b - (g \cdot b)_{x_2} a \\ &= g(b_{x_1} b - b_{x_2} a) + b(g_{x_1} b - g_{x_2} a) \\ &= -ga(a_{x_1} + b_{x_2}) + b(g_{x_1} b - g_{x_2} a) \end{aligned}$$

and

$$\begin{aligned} 0 &= -(g \cdot a)_{x_1} b + (g \cdot a)_{x_2} a \\ &= g(-a_{x_1} b + a_{x_2} a) - a(g_{x_1} b - g_{x_2} a) \\ &= -gb(a_{x_1} + b_{x_2}) - a(g_{x_1} b - g_{x_2} a). \end{aligned}$$

Therefore

$$\begin{bmatrix} -ga & b \\ -gb & -a \end{bmatrix} \begin{bmatrix} a_{x_1} + b_{x_2} \\ g_{x_1} b - g_{x_2} a \end{bmatrix} = 0.$$

It follows that $a_{x_1} + b_{x_2} = 0$ and $g_{x_1} b = g_{x_2} a$. □

LEMMA 21. $(g \cdot a)_{x_2} = (g \cdot b)_{x_1}$.

Proof. Let k be an integer number. Since $n_i^{-1} Df^{n_i+k} \to h_k$ in $C^1(\mathbb{T}^2, M_2(\mathbb{R}))$ and

$$h_k = \delta_k g_k \begin{bmatrix} -a \cdot b \circ f^k & -b \cdot b \circ f^k \\ a \cdot a \circ f^k & b \cdot a \circ f^k \end{bmatrix}$$

(by Lemma 19), we have

$$\begin{aligned} (g_k \cdot a \cdot b \circ f^k)_{x_2} &= -\delta_k \lim_{i \to \infty} \frac{1}{n_i} (f_1^{n_i+k})_{x_1 x_2} \\ &= -\delta_k \lim_{i \to \infty} \frac{1}{n_i} (f_1^{n_i+k})_{x_2 x_1} \\ &= (g_k \cdot b \cdot b \circ f^k)_{x_1} \end{aligned}$$

and

$$\begin{aligned} (g_k \cdot a \cdot a \circ f^k)_{x_2} &= \delta_k \lim_{i \to \infty} \frac{1}{n_i} (f_2^{n_i+k})_{x_1 x_2} \\ &= \delta_k \lim_{i \to \infty} \frac{1}{n_i} (f_2^{n_i+k})_{x_2 x_1} \\ &= (g_k \cdot b \cdot a \circ f^k)_{x_1}. \end{aligned}$$

Suppose that $e : \mathbb{T}^2 \to \mathbb{R}$ is a C^1-function satisfying

$$(g_k \cdot a \cdot e \circ f^k)_{x_2} = (g_k \cdot b \cdot e \circ f^k)_{x_1} \tag{12}$$

and $De \cdot v = 0$. Observe that the functions a and b satisfy these properties. Indeed, by Lemma 20,

$$Da \cdot v = [a_{x_1} \; a_{x_2}] \begin{bmatrix} -b \\ a \end{bmatrix} = -a_{x_1} b + a_{x_2} a = b_{x_2} b + a_{x_2} a = 0$$

and

$$Db \cdot v = [b_{x_1}\ b_{x_2}] \begin{bmatrix} -b \\ a \end{bmatrix} = -b_{x_1}b + b_{x_2}a = -b_{x_1}b - a_{x_1}a = 0.$$

Since $g_k/g_0 \circ f^k = g/g \circ f^k$, (12) shows that

$$(ga \cdot (g_0 g^{-1} e) \circ f^k)_{x_2} = (gb \cdot (g_0 g^{-1} e) \circ f^k)_{x_1}.$$

Hence

$$\begin{aligned}((gb)_{x_1} - (ga)_{x_2}) \cdot (g_0 g^{-1} e) \circ f^k &= g(-b((g_0 g^{-1} e) \circ f^k)_{x_1} + a((g_0 g^{-1} e) \circ f^k)_{x_2}) \\ &= gD((g_0 g^{-1} e) \circ f^k) \cdot v \\ &= D(g_0 g^{-1} e) \circ f^k \cdot Df^k \cdot (gv).\end{aligned}$$

Since $(g \cdot v) \circ f^k = (\varepsilon_1 \varepsilon)^k Df^k (g \cdot v)$ (see (7)), we have

$$\begin{aligned}((gb)_{x_1} - (ga)_{x_2}) \cdot (g_0 g^{-1} e) \circ f^k &= (\varepsilon_1 \varepsilon)^k (D(g_0 g^{-1} e) \cdot gv) \circ f^k \\ &= (\varepsilon_1 \varepsilon)^k (D(g_0 g^{-1}) \cdot egv + g_0 De \cdot v) \circ f^k \\ &= (\varepsilon_1 \varepsilon)^k (D(g_0 g^{-1}) \cdot egv) \circ f^k.\end{aligned}$$

Letting $c = a$ and $c = b$ we obtain

$$(gb)_{x_1} - (ga)_{x_2} = (\varepsilon_1 \varepsilon)^k (D(g_0 g^{-1}) \cdot g_0^{-1} g^2 v) \circ f^k.$$

Therefore, the function $|(gb)_{x_1} - (ga)_{x_2}|$ is $\mathcal{A}_f$-measurable. Since $\mathcal{A}_f$ is finite, the function $(gb)_{x_1} - (ga)_{x_2}$ is constant. However,

$$\int_{\mathbb{T}^2} ((gb)_{x_1} - (ga)_{x_2})\, d\lambda = 0.$$

Hence $(gb)_{x_1} = (ga)_{x_2}$. □

Proof of Theorem 13. By the previous lemmas, there exists $g \in C^{1+L}(\mathbb{T}^2, \mathbb{R})$ such that:

- $0 < c \leq g(\bar{x})$ for any $\bar{x} \in \mathbb{T}^2$;
- $(g \cdot u^T) \circ f\, Df = \varepsilon_1 (g \cdot u^T)$; and
- $(g \cdot b)_{x_1} = (g \cdot a)_{x_2}$, $a_{x_1} + b_{x_2} = 0$ and $g_{x_1} b = g_{x_2} a$.

It follows that there exists a C^{2+L}-function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$D\xi = [g \cdot a\ g \cdot b] = g \cdot u^T.$$

Consider the map

$$(x_1, x_2) \longmapsto \xi(x_1 + 1, x_2) - \xi(x_1, x_2).$$

Since its derivative is equal to zero, we see that it is constant. Similarly

$$(x_1, x_2) \longmapsto \xi(x_1, x_2 + 1) - \xi(x_1, x_2).$$

is constant. Therefore, ξ can be represented as

$$\xi(x_1, x_2) = p_1 x_1 + p_2 x_2 + \tilde{\xi}(x_1, x_2), \tag{13}$$

where $p_1, p_2 \in \mathbb{R}$ and $\tilde{\xi} : \mathbb{T}^2 \to \mathbb{R}$ is a C^2-function. Note that $p_1^2 + p_2^2 > 0$. Indeed, since $\tilde{\xi}$ is $\mathbb{Z}^2$-periodic, there exists $\bar{x}_0 \in \mathbb{T}^2$ such that $D\tilde{\xi}(\bar{x}_0) = 0$. Then $p_1 = \xi_{x_1}(\bar{x}_0) = (g \cdot a)(\bar{x}_0)$, $p_2 = \xi_{x_2}(\bar{x}_0) = (g \cdot b)(\bar{x}_0)$ and $p_1^2 + p_2^2 = g^2(\bar{x}_0) > 0$. Since $(g \cdot u^T) \circ f \cdot Df = \varepsilon_1 (g \cdot u^T)$, we have $D\xi \circ f \cdot Df = \varepsilon_1 D\xi$. Hence $D(\xi \circ f - \varepsilon_1 \xi) = 0$. Therefore there exists $\beta \in \mathbb{R}$ such that

$$\xi \circ f = \varepsilon_1 \xi + \beta. \tag{14}$$

Represent f as

$$f(x_1, x_2) = (a_{11}x_1 + a_{12}x_2 + \widetilde{f_1}(x_1, x_2), a_{21}x_1 + a_{22}x_2 + \widetilde{f_2}(x_1, x_2)),$$

where $[a_{ij}]_{i,j=1,2} \in GL_2(\mathbb{Z})$ and $\widetilde{f_1}, \widetilde{f_2} : \mathbb{T}^2 \to \mathbb{R}$. From (14),

$$p_1 a_{11} + p_2 a_{21} = \varepsilon_1 p_1 \quad \text{and} \quad p_1 a_{12} + p_2 a_{22} = \varepsilon_1 p_2.$$

Observe that there exists a real number $d \neq 0$ such that

$$p_1 d, p_2 d \in \mathbb{Z} \quad \text{and} \quad \gcd(p_1 d, p_2 d) = 1.$$

If one of the numbers p_1, p_2 is equal to zero, then $d = 1/(p_1 + p_2)$. If $p_1, p_2 \neq 0$, then set $d_1 = p_1/p_2$. Hence

$$a_{11} + \frac{1}{d_1} a_{21} = \varepsilon_1 \quad \text{and} \quad d_1 a_{12} + a_{22} = \varepsilon_1.$$

Note that d_1 is rational. Indeed, suppose that d_1 is irrational. Then $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] = \varepsilon_1 \mathrm{Id}$. Therefore f^2 can be represented as

$$f^2(x_1, x_2) = (x_1 + \psi_1(x_1, x_2), x_2 + \psi_2(x_1, x_2)),$$

where $\psi_1, \psi_2 : \mathbb{T}^2 \to \mathbb{R}$ are C^3-functions. Then

$$f^{2n}(x_1, x_2) = \left(x_1 + \sum_{k=0}^{n-1} \psi_1(f^{2k}(x_1, x_2)), x_2 + \sum_{k=0}^{n-1} \psi_2(f^{2k}(x_1, x_2))\right)$$

and

$$\frac{1}{n}(f^{2n} - \mathrm{Id}) = \left(\frac{1}{n} \sum_{k=0}^{n-1} \psi_1 \circ f^{2k}, \frac{1}{n} \sum_{k=0}^{n-1} \psi_2 \circ f^{2k}\right). \tag{15}$$

Since the sequence $\{n^{-1} Df^{2n}\}_{n \in \mathbb{N}}$ is bounded in $C^2(\mathbb{T}^2, M_2(\mathbb{R}))$ and $0 < 2c \leq \|n^{-1} Df^{2n}(\bar{x})\| \leq 2C$, there exists an increasing sequence $\{m_k\}_{k \in \mathbb{N}}$ of naturals and $\psi \in C^1(\mathbb{T}^2, \mathbb{R}^2)$ such that

$$\frac{1}{m_k}(f^{2m_k} - \mathrm{Id}) \to \psi, \quad \frac{1}{m_k}(Df^{2m_k} - \mathrm{Id}) \to D\psi$$

uniformly and $0 < 2c \leq \|D\psi(\bar{x})\|$. From (15), ψ is $\mathcal{A}_{f^2}$-measurable. By Lemma 10, the σ-algebra $\mathcal{A}_{f^2}$ is finite. Hence ψ is constant. This gives $D\psi = 0$, which contradicts the fact that $0 < 2c \leq \|D\psi(\bar{x})\|$ for any $\bar{x} \in \mathbb{T}^2$. Therefore d_1 is rational. Let $d_1 = p/q$ where $p \in \mathbb{Z}, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. Set $d = p/p_1$. Then $p_1 d = p$ and $p_2 d = q$.

Next note that $\varepsilon_1 = 1$. Suppose, contrary to our claim, that $\varepsilon_1 = -1$. Let $\widehat{\xi} : \mathbb{T}^2 \to S^1$ be given by $\widehat{\xi} = \exp 2\pi i \xi / d$. From (14), $\widehat{\xi} \circ f = e^{2\pi i \beta / d}\, \overline{\widehat{\xi}}$. Hence

$$\widehat{\xi} \circ f^2 = e^{2\pi i \beta / d}\, \overline{\widehat{\xi} \circ f} = \widehat{\xi}.$$

Since $\widehat{\xi}$ is $\mathcal{A}_{f^2}$-measurable and the σ-algebra $\mathcal{A}_{f^2}$ is finite, it follows that $\widehat{\xi}$ is constant, hence that ξ is constant, which contradicts the fact that $0 < c \le \|D\xi(\bar{x})\|$ for any $\bar{x} \in \mathbb{T}^2$.

However, $b_{x_1} = a_{x_2}$. Since

$$aa_{x_1} + ba_{x_2} = aa_{x_1} + bb_{x_1} = 0,$$
$$-ba_{x_1} + aa_{x_2} = bb_{x_2} + aa_{x_2} = 0,$$

we have $Da = 0$ and similarly $Db = 0$. Consequently a and b are constants and g satisfies the partial linear equation with constant coefficients $g_{x_1} b - g_{x_2} a = 0$. It follows that there exists a strictly increasing C^2-function $w : \mathbb{R} \to \mathbb{R}$ such that

$$g(x_1, x_2) = Dw(ax_1 + bx_2).$$

Hence

$$\xi_{x_1}(x_1, x_2) = Dw(ax_1 + bx_2)a \quad \text{and} \quad \xi_{x_2}(x_1, x_2) = Dw(ax_1 + bx_2)b.$$

Therefore we can assume that $\xi(x_1, x_2) = w(ax_1 + bx_2)$. It follows that

$$w(x + a) = p_1(x/a + 1) + \tilde{\xi}(x/a + 1, 0) = w(x) + p_1,$$
$$w(x + b) = p_2(x/b + 1) + \tilde{\xi}(0, x/b + 1) = w(x) + p_2,$$

by (13). Consequently

$$w(x + qa) = qp_1 = pp_2 = w(x + pb)$$

and $qa = pb$. Define $\tilde{w}(x) = dw(bx/q)$. Then $dw(ax_1 + bx_2) = \tilde{w}(px_1 + qx_2)$ and $\tilde{w}(x + 1) = \tilde{w}(x) + 1$. Moreover,

$$\tilde{w}(pf_1(\overline{x}) + qf_2(\overline{x})) = \tilde{w}(px_1 + qx_2) + \beta',$$

by (14). Let r, s be integer numbers such that $ps - qr = 1$. Consider the group automorphism $B : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $B(x_1, x_2) = (px_1 + qx_2, rx_1 + sx_2)$. Let $\check{f} = B \circ f \circ B^{-1}$ and $\pi_i : \mathbb{T}^2 \to \mathbb{T}$ be the projection on the ith coordinate for $i = 1, 2$. Then

$$\begin{aligned}\tilde{w}(\check{f}_1(x_1, x_2)) &= \tilde{w}(p\, f_1 \circ B^{-1}(x_1, x_2) + q\, f_2 \circ B^{-1}(x_1, x_2)) \\ &= \tilde{w}(p\, \pi_1 \circ B^{-1}(x_1, x_2) + q\, \pi_2 \circ B^{-1}(x_1, x_2)) + \beta' \\ &= \tilde{w}(x_1) + \beta'.\end{aligned}$$

Therefore, $\check{f}_1$ depends only on the first variable. Then

$$D\check{f} = \begin{bmatrix} \dfrac{\partial \check{f}_1}{\partial x_1} & 0 \\ \dfrac{\partial \check{f}_2}{\partial x_1} & \dfrac{\partial \check{f}_2}{\partial x_2} \end{bmatrix} \quad \text{and} \quad \frac{\partial \check{f}_1}{\partial x_1}\frac{\partial \check{f}_2}{\partial x_2} = \det D\check{f} = \varepsilon.$$

Since $(\partial \check{f}_2/\partial x_2)(x_1, x_2) = \varepsilon/(\partial \check{f}_1/\partial x_1)(x_1, 0)$, there exists a C^3-function $\varphi : \mathbb{T} \to \mathbb{T}$ such that

$$\check{f}_2(x_1, x_2) = \frac{\varepsilon}{(\partial \check{f}_1/\partial x_1)(x_1, 0)} x_2 + \varphi(x_1)$$

and $\varepsilon/(\partial \check{f}_1/\partial x_1)(x_1, 0)$ is an integer constant. As the map $\mathbb{T} \ni x \longmapsto \check{f}_1(x, 0) \in \mathbb{T}$ is continuous and increasing, it follows that $(\partial \check{f}_1/\partial x_1)(x_1, 0) = 1$. Therefore

$$\check{f}(x_1, x_2) = (x_1 + \alpha, \varepsilon x_2 + \varphi(x_1)),$$

where α is irrational, by the ergodicity of f. Next note that $\varepsilon = 1$. Indeed, suppose that $\varepsilon = -1$. Then

$$\frac{1}{2n} D\check{f}^{2n}(x_1, x_2) = \begin{bmatrix} \dfrac{1}{2n} & 0 \\ \dfrac{1}{2n} \displaystyle\sum_{k=0}^{n-1} (D\varphi(x_1 + \alpha + 2k\alpha) - D\varphi(x_1 + 2k\alpha)) & \dfrac{1}{2n} \end{bmatrix}.$$

By the ergodic theorem,

$$\frac{1}{2n} \sum_{k=0}^{n-1} (D\varphi(x_1 + \alpha + 2k\alpha) - D\varphi(x_1 + 2k\alpha)) \to \frac{1}{2} \int_{\mathbb{T}} (D\varphi(x + \alpha) - D\varphi(x))\, dx = 0$$

uniformly, which contradicts the fact that f has linear growth of the derivative. □

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