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A note on quasi-similarity of Koopman operators

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Abstract

Answering a question of A. Vershik we construct two non-weakly isomorphic ergodic automorphisms for which the associated unitary (Koopman) representations are Markov quasi-similar. We also discuss metric invariants of Markov quasi-similarity in the class of ergodic automorphisms.

1. *Introduction*

Markov operators appear in the classical ergodic theory in the context of joinings, see the monograph [**7**]. Indeed, assume that T_i is an ergodic automorphism of a standard probability Borel space $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$. Consider λ a *joining* of T_1 and T_2, i.e. a $T_1 \times T_2$-invariant probability measure on $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ with the marginals μ_1 and μ_2 respectively. Then the operator $\Phi_\lambda : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ determined by

$$\langle \Phi_\lambda f_1, f_2 \rangle_{L^2(X_2,\mathcal{B}_2,\mu_2)} = \langle f_1 \otimes \mathbf{1}_{X_2}, \mathbf{1}_{X_1} \otimes f_2 \rangle_{L^2(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2, \lambda)} \tag{1.1}$$

is Markov (i.e. it is a linear contraction which preserves the cone of non-negative functions and $\Phi_\lambda \mathbf{1} = \mathbf{1} = \Phi_\lambda^* \mathbf{1}$) and moreover

$$\Phi_\lambda \circ U_{T_1} = U_{T_2} \circ \Phi_\lambda, \tag{1.2}$$

where $U_{T_i} : L^2(X_i, \mathcal{B}_i, \mu_i) \to L^2(X_i, \mathcal{B}_i, \mu_i)$ stands for the associated unitary operator: $U_{T_i} f = f \circ T_i$ for $f \in L^2(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, which is often called a *Koopman operator*. In fact, each Markov operator $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ satisfying the equivariance property (1.2) is of the form Φ_λ for a unique joining λ of T_1 and T_2 (see e.g. [**17**], [**24**]). Markov operators corresponding to ergodic joinings are called *indecomposable*.

In order to classify dynamical systems one usually considers the *measure-theoretic isomorphism*, i.e. the equivalence given by the existence of an invertible map $S : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ for which $S \circ T_1 = T_2 \circ S$. The measure-theoretic (metric) isomorphism implies spectral equivalence of the Koopman operators U_{T_1} and U_{T_2}; indeed, $U_{S^{-1}}$ (where $U_{S^{-1}} f_1 = f_1 \circ S^{-1}$ for $f_1 \in L^2(X_1, \mathcal{B}_1, \mu_1)$) provides such an equivalence. The converse does not hold, see e.g. [**1**]; we also recall that all Bernoulli shifts are spectrally equivalent while the entropy classify them measure-theoretically [**19**]. One may ask whether there can be some other natural classification of dynamical systems which lies in between metric and spectral equivalence.

Given $(X, \mathcal{B}, \mu)$ a standard probability Borel space, following [**26**], each probability measure on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ with both marginals μ is called a *polymorphism*. Regarding automorphisms of $(X, \mathcal{B}, \mu)$ as the corresponding graph measures, in [**26**], Vershik originates a new theory – the theory of polymorphisms – in which polymorphisms are analogues of automorphisms of $(X, \mathcal{B}, \mu)$. Since, in view of (1.1), there is a one-to-one correspondence between polymorphisms and Markov operators of $L^2(X, \mathcal{B}, \mu)$, as the corresponding equivalence (borrowed for operator theory, see below) Vershik has chosen Markov quasi-similarity. In particular, Vershik proposed

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to study this new equivalence between polymorphisms and automorphisms, and even between automorphisms themselves.

Recall that if A_i is a bounded linear operator of a Hilbert space H_i, $i = 1, 2$, and if there is a bounded linear operator $V : H_1 \to H_2$ whose range is dense and which intertwines A_1 and A_2, this is $V \circ A_1 = A_2 \circ V$, then A_2 is said to be a *quasi-image* of A_1 (see [**4**]). By duality, A_2 is a quasi-image of A_1 if and only if there exists a $1-1$ bounded linear operator $W : H_2 \to H_1$ intertwining A_2 and A_1. If also A_1 is a quasi-image of A_2 then the two operators are called *quasi-similar*. Recall also that two operators A_1 and A_2 are *quasi-affine* if there exists a $1-1$ bounded linear operator $V : H_1 \to H_2$ with dense range intertwining A_1 and A_2. In general, the notion of quasi-affinity is stronger than quasi-similarity (see however Remark 2.2 below).

Assume additionally that A_i is a Markov operator of $H_i = L^2(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$. If A_2 is a quasi-image of A_1 and we additionally require $V : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ to be a Markov operator then A_2 is said to be a *Markov quasi-image* of A_1. If additionally A_1 is a Markov quasi-image of A_2 then the two operators are called *Markov quasi-similar*. Operators A_1 and A_2 are *Markov quasi-affine* if there exists a $1-1$ Markov operator V between the corresponding L^2-spaces with dense range intertwining A_1 and A_2.

Notice that each Koopman operator is also a Markov operator. It is known (see e.g. [**15**], [**26**]) that if an intertwining Markov operator $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ is unitary then it has to be of the form U_S where S provides a measure-theoretic isomorphism. On the other hand the quasi-similarity of unitary operators implies their spectral equivalence (see Section 2 below). Therefore, Markov quasi-similarity lies in between the spectral and measure-theoretic equivalence of dynamical systems. One of questions raised by Vershik in [**26**] is the following:

Do there exist two automorphisms that are not isomorphic but are Markov quasi-similar? (1.3)

In order to answer this question notice that any weakly isomorphic automorphisms (see [**25**]) T_1 and T_2 are automatically Markov quasi-similar; indeed, the weak isomorphism means that there are π_1 and π_2 which are homomorphisms between T_1 and T_2 and T_2 and T_1 respectively, then $U^*_{\pi_1}$ and $U^*_{\pi_2}$ yield Markov quasi-similarity of T_1 and T_2. Hence, if T_1 and T_2 are weakly isomorphic but not isomorphic, we obtain the positive answer to the question (1.3). The first examples of weakly isomorphic but not isomorphic systems were given by Polit in [**21**]. For further examples we refer the reader to [**12**], [**13**], [**23**], including the case of K-automorphisms [**8**]. It follows that the notion of Markov quasi-similarity has to be considered as an interesting refinement of the notion of weak isomorphism, and in Vershik's question (1.3) we have to replace "not isomorphic" by "not weakly isomorphic".

The main aim of this note is to answer positively this modified question (1.3) (see Proposition 4.4 below). We would like to emphasize that despite a spectral flavor of the definition, Markov quasi-similarity is far from being the same as spectral equivalence. For example, partly answering Vershik's question raised at a seminar at Penn State University in 2004 whether entropy is an invariant of Markov quasi-similarity, we show that zero entropy as well as K-property are invariants of Markov quasi-similarity of automorphisms, while they are not invariants of spectral equivalence of the corresponding unitary operators. These facts and related problems will be discussed in Sections 5-7.

2. *Quasi-similarity of unitary operators implies their unitary equivalence*

Assume that U is a unitary operator of a separable Hilbert space H. Given $x \in H$ by $\mathbb{Z}(x)$ we denote the *cyclic space generated by* x, i.e. $\mathbb{Z}(x) = \overline{\mathrm{span}}\{U^n x : n \in \mathbb{Z}\}$. We will use a similar notation $\mathbb{Z}(y_1, \ldots, y_k)$ for the smallest closed U-invariant subspace containing y_i, $i = 1, \ldots, k$. Denote by $\mathbb{T}$ the (additive) circle. Then the Fourier transform of the (positive) measure σ_x –

called the *spectral measure of* x – is given by

$$\widehat{\sigma}_x(n) := \int_{\mathbb{T}} e^{2\pi int}\, d\sigma_x(t) = \langle U^n x, x\rangle \text{ for each } n \in \mathbb{Z}.$$

Similarly the sequence $(\langle U^n x, y\rangle)_{n\in\mathbb{Z}}$ is the Fourier transform of the (complex) spectral measure $\sigma_{x,y}$ of x and y. Given a spectral measure σ we denote

$$H_\sigma = \{x \in H :\ \sigma_x \ll \sigma\}.$$

Then H_σ is a closed U-invariant subspace called a *spectral subspace* of H.

It follows from Spectral Theorem for unitary operators (see e.g. [**11**] or [**20**]) that there is a decomposition

$$H = H_{\sigma_1} \oplus H_{\sigma_2} \oplus \dots \tag{2.1}$$

into spectral subspaces such that for each $i \geq 1$

$$H_{\sigma_i} = \bigoplus_{k=1}^{n_i} \mathbb{Z}(x_k^{(i)}),$$

where $\sigma_i \equiv \sigma_{x_1^{(i)}} \equiv \sigma_{x_2^{(i)}} \equiv \dots$ (n_i can be infinity), and $\sigma_i \perp \sigma_j$ for $i \neq j$. The class σ_U of all finite measures equivalent to the sum $\sum_{i\geq 1} \sigma_i$ is then called the *maximal spectral type of* U. Another important invariant of U is the spectral multiplicity function $M_U : \mathbb{T} \to \{1, 2, \dots\} \cup \{\infty\}$ (see [**11**], [**20**]) which is defined σ-a.e., where σ is any measure belonging to the maximal spectral type of U. Note that decomposition (2.1) is far from being unique but if

$$H = \bigoplus_{i=1}^{\infty} H_{\sigma_i'},\quad H_{\sigma_i'} = \bigoplus_{k=1}^{n_i'} \mathbb{Z}(y_k^{(i)})$$

is another decomposition (2.1) in which $\sigma_i \equiv \sigma_i'$, $i \geq 1$, then $n_i = n_i'$ for $i \geq 1$. Recall that the essential supremum m_U of M_U (called the *maximal spectral multiplicity of* U) is equal to

$$\inf\{m \geq 1 :\ \mathbb{Z}(y_1, \dots, y_m) = H \quad \text{for some}\ \ y_1, \dots, y_m \in H\}; \tag{2.2}$$

if there is no "good" m, them $m_U = \infty$.

Assume that U_i is a unitary operator of a separable Hilbert space H_i, $i = 1, 2$. Let $V : H_1 \to H_2$ be a bounded linear operator which intertwines U_1 and U_2. Then for each $n \in \mathbb{Z}$ and $x_1 \in H_1$

$$\langle U_2^n V x_1, V x_1\rangle = \langle U_1^n x_1, V^* V x_1\rangle,$$

so by elementary properties of spectral measures

$$\sigma_{V x_1} = \sigma_{x_1, V^* V x_1} \ll \sigma_{x_1}. \tag{2.3}$$

Assuming additionally that $\mathrm{Im}(V)$ is dense, an immediate consequence of (2.3) is that the maximal spectral type of a quasi-image of U_1 is absolutely continuous with respect to σ_{U_1}. It is also clear that given $y_1^{(1)}, \dots, y_m^{(1)} \in H_1$ we have

$$\overline{V(\mathbb{Z}(y_1^{(1)}, \dots, y_m^{(1)}))} = \mathbb{Z}(V y_1^{(1)}, \dots, V y_m^{(1)}).$$

This in turn implies that the maximal spectral multiplicity of a quasi-image of U_1 is at most m_{U_1}.

PROPOSITION 2.1. *If U_1 and U_2 are quasi-similar then they are spectrally equivalent.*

Proof. Assume that $V : H_1 \to H_2$ and $W : H_2 \to H_1$ intertwine U_1 and U_2 and have dense ranges. In view of (2.3) both operators U_1 and U_2 have the same maximal spectral types.

Consider a decomposition (2.1) for U_1: $H_1 = \bigoplus_{i\geq 1} H_{\sigma_i^{(1)}}$ and let $F_i := \overline{V(H_{\sigma_i^{(1)}})}$ for $i \geq 1$. The subspaces F_i are obviously U_2-invariant and let $\sigma_i^{(2)}$ ($n_i^{(2)}$) denote the maximal spectral type (the maximal spectral multiplicity) of U_2 on F_i. It follows from (2.3) that $\sigma_i^{(2)} \ll \sigma_i^{(1)}$ for $i \geq 1$ and $\sigma_i^{(2)}, \sigma_j^{(2)}$ are mutually singular (in particular, $F_i \perp F_j$) whenever $i \neq j$. Moreover, $n_i^{(2)} \leq n_i^{(1)}$, $i \geq 1$. Since V has dense range, $H_2 = \bigoplus_{i\geq 1} F_i$. It follows that (up to equivalence of measures) $\sum_{i\geq 1} \sigma_2^{(i)}$ is the maximal spectral type of U_2 hence it is equivalent to $\sum_{i\geq 1} \sigma_i^{(1)}$ and therefore $\sigma_i^{(1)} \equiv \sigma_i^{(2)}$ for $i \geq 1$. The same reasoning applied to the decomposition $H_2 = \bigoplus_{i\geq 1} F_i$ and W shows that $H_1 = \bigoplus_{i\geq 1} \overline{W(F_i)}$ and the maximal spectral type of U_1 on $\overline{W(F_i)}$ is absolutely continuous with respect to $\sigma_i^{(2)} \equiv \sigma_i^{(1)}$, $i \geq 1$. It follows that $\overline{W(F_i)} = H_{\sigma_i^{(1)}}$ for all $i \geq 1$. In particular, we have proved that $n_i^{(2)} = n_i^{(1)}$ but we need to show that on F_i the multiplicity is uniform. Suppose this is not the case, i.e. that for some measure $\eta \ll \sigma_i^{(2)}$ we have

$$F_i = \mathbb{Z}(z_1) \oplus \ldots \oplus \mathbb{Z}(z_r) \oplus F_i',$$

where for $j = 1, \ldots r$, $\sigma_{z_j} = \eta$, $1 \leq r < n_i^{(2)}$ and the maximal spectral type of U_2 on F_i' is orthogonal to η. We have

$$H_{\sigma_i^{(1)}} = \overline{W(F_i)} = G_i \oplus \overline{W(F_i')},$$

where $G_i = \overline{W(\mathbb{Z}(z_1) \oplus \ldots \oplus \mathbb{Z}(z_r))}$ and the maximal spectral types on G_i, say $\tau(\ll \eta)$, and $\overline{W(F_i')}$ are mutually singular. It follows that the multiplicity of τ is at most r, which is a contradiction since all measures absolutely continuous with respect to $\sigma_i^{(1)}$ have multiplicity $n_i^{(1)}$. □

REMARK 2.2. Literally speaking, the notion of quasi-similarity is weaker than the notion of quasi-affinity. Proposition 3.4 in **[4]** tells us that quasi-affine unitary operators are unitarily equivalent. Hence Proposition 2.1 shows in fact that for unitary operators quasi-similarity and quasi-affinity are equivalent notions.

It is not clear (see Section 7) whether the notions of Markov quasi-similarity and Markov quasi-affinity of Koopman operators coincide.

3. *A convolution operator in* $l^2(\mathbb{Z})$

In this section we produce a sequence in $l^2(\mathbb{Z})$ which will be used to construct a Markov quasi-affinity between two non-weakly isomorphic automorphisms in Section 4.

Denote by $l_0(\mathbb{Z})$ the subspace of $l^2(\mathbb{Z})$ of complex sequences $\bar{x} = (x_n)_{n\in\mathbb{Z}}$ such that $\{n \in \mathbb{Z} : x_n \neq 0\}$ is finite.

PROPOSITION 3.1. *There exists a nonnegative sequence* $\bar{a} = (a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ *such that* $\sum_{n\in\mathbb{Z}} a_n = 1$ *and*

$$\text{for every } \bar{x} = (x_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z}) \text{ if } \bar{a} * \bar{x} \in l_0(\mathbb{Z}) \text{ then } \bar{x} = \bar{0}. \tag{3.1}$$

Each element $\bar{y} \in l^2(\mathbb{Z})$ is an L^2-function on $\mathbb{Z}$ and its Fourier transform is a function $h \in L^2(\mathbb{T})$ for which $\hat{h}(n) = y_n$ for all $n \in \mathbb{Z}$. Moreover, the convolution of l^2-sequences corresponds to the pointwise multiplication of L^2-functions on the circle. It follows that in order to find the required sequence $\bar{a}$, it suffices to find a function $f \in L^2(\mathbb{T})$ such that

– $a_n = \hat{f}(n) \geq 0$, $\sum_{n\in\mathbb{Z}} a_n = 1$;

– for every $g \in L^2(\mathbb{T})$, if $f \cdot g = 0$ then $g = 0$;
– for every non-zero trigonometric polynomial P, if $P = f \cdot g$ then $g \notin L^2(\mathbb{T})$.
This is done below.

LEMMA 3.2. *If $f : [0,1] \to \mathbb{R}_+$ is a convex C^2-function such that $f(1-x) = f(x)$ for all $x \in [0,1]$ then $\hat{f}(n) \geq 0$ for all $n \in \mathbb{Z}$.*

Proof. By assumption, $f''(x) \geq 0$ for all $x \in [0,1]$. Using integration by parts twice, for $n \neq 0$ we obtain

$$\begin{aligned}\hat{f}(n) &= \int_0^1 f(x)e^{-2\pi i n x}dx = \int_0^1 f(x)\cos(2\pi n x)\,dx = \frac{1}{2\pi n}\int_0^1 f(x)\,d\sin(2\pi n x)\\ &= -\frac{1}{2\pi n}\int_0^1 f'(x)\sin(2\pi n x)\,dx = \frac{1}{4\pi^2 n^2}\int_0^1 f'(x)\,d\cos(2\pi n x)\\ &= \frac{1}{4\pi^2 n^2}\left[f'(1) - f'(0) - \int_0^1 f''(x)\cos(2\pi n x)\,dx\right]\\ &\geq \frac{1}{4\pi^2 n^2}\left[f'(1) - f'(0) - \int_0^1 |f''(x)\cos(2\pi n x)|\,dx\right]\\ &\geq \frac{1}{4\pi^2 n^2}\left[f'(1) - f'(0) - \int_0^1 f''(x)\,dx\right] = 0.\end{aligned}$$

□

Proof of Proposition 3.1. Let us consider $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{|x-1/2|}+2} & \text{if} \quad x \neq 1/2 \\ 0 & \text{if} \quad x = 1/2. \end{cases}$$

Since $f''(x) \geq 0$ for $x \in [0,1]$, by Lemma 3.2, $a_n = \hat{f}(n) \geq 0$. As $f : \mathbb{T} \to \mathbb{R}$ is a continuous function of bounded variation,

$$1 = f(0) = \sum_{n\in\mathbb{Z}} a_n.$$

Since $f(x) \neq 0$ for $x \neq 1/2$, if $f \cdot g = 0$ for some $g \in L^2(\mathbb{T})$ then $g = 0$.

Suppose, contrary to our claim, that there exist $g \in L^2(\mathbb{T})$ and a non-zero trigonometric polynomial P such that $f \cdot g = P$. Recall that for every $m \geq 0$ we have $\int_0^1 e^{1/x}x^m\,dx = +\infty$, hence $\int_0^1 (e^{1/x}x^m)^2\,dx = +\infty$. Since P is a non-zero analytic function, there exists $m \geq 0$ such that $P^{(m)}(1/2) \neq 0$ and $P^{(k)}(1/2) = 0$ for $0 \leq k < m$. By Taylor's formula, there exist $C > 0$ and $0 < \delta < 1/2$ such that $|P(x + 1/2)| \geq C|x|^m$ for $x \in [-\delta, \delta]$. It follows that

$$\begin{aligned}\int_{\mathbb{T}} |g(x)|^2\,dx &\geq \int_{1/2}^{1/2+\delta} |P(x)|^2/f(x)^2\,dx = \int_0^\delta |P(x+1/2)|^2/f(x+1/2)^2\,dx\\ &\geq \int_0^\delta (Cx^m e^{1/x})^2\,dx = +\infty,\end{aligned}$$

and hence $g \notin L^2(\mathbb{T})$ which completes the proof. □

4. *Two non-weakly isomorphic automorphisms which are Markov quasi-similar*

Let T be an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Assume that G is a compact metric Abelian group with Haar measure λ_G. A measurable function $\varphi : X \to G$ is called a *cocycle*. Using the cocycle we can define a *group extension* T_φ of T which acts on $(X \times G, \mathcal{B}\otimes\mathcal{B}(G), \mu\otimes\lambda_G)$ by the formula $T_\varphi(x,g) = (Tx, \varphi(x)+g)$.

We will first take $\varphi : X \to \mathbb{Z}_2 := \{0,1\}$ so that the group extension T_φ is ergodic. Then assume that we can find S acting on $(X,\mathcal{B},\mu)$, $ST = TS$, such that if we put $G = \mathbb{Z}_2^{\mathbb{Z}}$ and define

$$\psi : X \to G, \;\; \psi(x) = (\ldots, \varphi(S^{-1}x), \overset{0}{\varphi(x)}, \varphi(Sx), \varphi(S^2x), \ldots)$$

then T_ψ is ergodic as well (see [**13**] for concrete examples of T, φ and S fulfilling our requirements). Put now $T_1 = T_\psi$ and let us take a factor T_2 of T_1 obtained by "forgetting" the first $\mathbb{Z}_2$-coordinate. In other words on $(X\times\mathbb{Z}_2^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{Z}_2^{\mathbb{Z}}})$ we consider two automorphisms

$$T_1(x,\underline{i}) = (Tx, \ldots, i_{-1}+\varphi(S^{-1}x), \overset{0}{i_0+\varphi(x)}, i_1+\varphi(Sx), i_2+\varphi(S^2x), \ldots),$$

$$T_2(x,\underline{i}) = (Tx, \ldots, i_{-1}+\varphi(S^{-1}x), \overset{0}{i_0+\varphi(x)}, i_1+\varphi(S^2x), i_2+\varphi(S^3x), \ldots),$$

where $\underline{i} = (\ldots, i_{-1}, \overset{0}{i_0}, i_1, i_2, \ldots)$. Define $I_n : X\times\mathbb{Z}_2^{\mathbb{Z}} \to X\times\mathbb{Z}_2^{\mathbb{Z}}$ by putting

$$I_n(x,\underline{i}) = (S^n x, \ldots, i_{n-1}, \overset{0}{i_n}, i_{n+2}, i_{n+3}, \ldots).$$

Then I_n is measure-preserving and $I_n\circ T_1 = T_2\circ I_n$. Therefore

$$U_{T_1}\circ U_{I_n} = U_{I_n}\circ U_{T_2} \tag{4.1}$$

with U_{I_n} being an isometry (which is not onto) and

$$\begin{aligned} &U^*_{I_n}F(x,\underline{i})\\ &= \frac{1}{2}\left(F(S^{-n}x, \ldots, \overset{0}{i_{-n}}, \ldots, \overset{n}{i_0}, 0, i_1, \ldots) + F(S^{-n}x, \ldots, \overset{0}{i_{-n}}, \ldots, \overset{n}{i_0}, 1, i_1, \ldots)\right). \end{aligned}$$

Let $\bar{a} = (a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ be a nonnegative sequence such that $\sum_{n\in\mathbb{Z}} a_n = 1$ and (3.1) holds. Let $J : L^2(X\times\mathbb{Z}_2^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{Z}_2^{\mathbb{Z}}}) \to L^2(X\times\mathbb{Z}_2^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{Z}_2^{\mathbb{Z}}})$ stand for the Markov operator defined by

$$J = \sum_{n\in\mathbb{Z}} a_n U_{I_n}.$$

In view of (4.1), J intertwines U_{T_1} and U_{T_2}.

Denote by Fin the set of finite nonempty subsets of $\mathbb{Z}$. The set Fin may be identified with the group of characters of the group $\mathbb{Z}_2^{\mathbb{Z}}$. Let us consider two operations on Fin:

$$\mathrm{G}(A) = \{s\in A:\ s\le 0\}\cup\{s+1:\ s\in A, s>0\} \text{ for } A\in Fin;$$

$$\mathrm{G}^{-1}(B) = \{s\in B:\ s\le 0\}\cup\{s-1:\ s\in B, s>1\} \text{ for } B\in Fin \text{ with } 1\notin B.$$

Of course, $\mathrm{G}^{-1}(\mathrm{G}(A)) = A$ and $\mathrm{G}(\mathrm{G}^{-1}(B)) = B$. Let $\sim$ stand for the equivalence relation in Fin defined by $A\sim B$ if $A = B+n$ for some $n\in\mathbb{Z}$. Denote by Fin_0 a fundamental domain for this relation.

Lemma 4.1. *J has trivial kernel.*

Proof. Each $F\in L^2(X\times\mathbb{Z}_2^{\mathbb{Z}}, \mu\otimes\lambda_{\mathbb{Z}_2^{\mathbb{Z}}})$ can be written as

$$F(x,\underline{i}) = \sum_{A\in Fin} f_A(x)(-1)^{A(\underline{i})}, \text{ where } A(\underline{i}) = \sum_{s\in A} i_s.$$

Note that $\sum_{A\in Fin}\|f_A\|^2_{L^2(X,\mu)}=\|F\|^2_{L^2(X\times\mathbb{Z}_2^{\mathbb{Z}},\mu\otimes\lambda_{\mathbb{Z}_2^{\mathbb{Z}}})}$. Since

$$U_{I_n}\left(f_A\otimes(-1)^{A(\cdot)}\right)(x,\underline{i})=\left(f_A\otimes(-1)^{A(\cdot)}\right)(I_n(x,\underline{i}))=f_A(S^nx)(-1)^{(\mathrm{G}(A)+n)(\underline{i})},$$

we have

$$JF(x,\underline{i})=\sum_{n\in\mathbb{Z}}\sum_{A\in Fin}a_nf_A(S^nx)(-1)^{(\mathrm{G}(A)+n)(\underline{i})}.$$

Notice that $n+1\notin\mathrm{G}(A)+n$. To reverse the roles played by A and $\mathrm{G}(A)+n$ note that if $B\in Fin$ and $n+1\notin B$ then the set $\mathrm{G}^{-1}(B-n)$ is the unique set such that $\mathrm{G}(\mathrm{G}^{-1}(B-n)))+n=B$. It follows that

$$JF(x,\underline{i})=\sum_{B\in Fin}\sum_{n\in\mathbb{Z},n+1\notin B}a_nf_{\mathrm{G}^{-1}(B-n)}(S^nx)(-1)^{B(\underline{i})}=\sum_{B\in Fin}\widetilde{F}_B(x)(-1)^{B(\underline{i})},$$

where $\widetilde{F}_B(x)=\sum_{n\in\mathbb{Z},n+1\notin B}a_nf_{\mathrm{G}^{-1}(B-n)}(S^nx)$. For every $B\in Fin_0$ and $x\in X$ we define $\xi^B(x)=(\xi^B_n(x))_{n\in\mathbb{Z}}$ by setting

$$\xi^B_{-n}(x)=\begin{cases} f_{\mathrm{G}^{-1}(B-n)}(S^nx) & \text{if} \quad n+1\notin B\\ 0 & \text{if} \quad n+1\in B.\end{cases}$$

Therefore, for $k\in\mathbb{Z}$

$$\begin{aligned}\widetilde{F}_{B+k}(x)&=\sum_{n\in\mathbb{Z},n+1\notin B+k}a_nf_{\mathrm{G}^{-1}(B-n+k)}(S^nx)\\ &=\sum_{n\in\mathbb{Z},(n-k)+1\notin B}a_nf_{\mathrm{G}^{-1}(B-(n-k))}(S^{-(k-n)}(S^kx))\\ &=\sum_{n\in\mathbb{Z}}a_n\xi^B_{k-n}(S^kx)=[\bar{a}*\left(\xi^B(S^kx)\right)]_k.\end{aligned}$$

Suppose that $J(F)=0$. It follows that given $k\in\mathbb{Z}$ and $B\in Fin_0$ we have $[\bar{a}*\left(\xi^B(S^kx)\right)]_k=\widetilde{F}_{B+k}(x)=0$ for μ-a.e. $x\in X$, whence a.s. we also have $[\bar{a}*\left(\xi^B(x)\right)]_k=0$. Letting k run through $\mathbb{Z}$ we obtain that $\bar{a}*\left(\xi^B(x)\right)=\bar{0}$ for μ-a.e. $x\in X$. On the other hand $\xi^B(x)\in l^2(\mathbb{Z})$ for almost every $x\in X$. In view of (3.1), $\xi^B(x)=\bar{0}$ for every $B\in Fin_0$ and for a.e. $x\in X$, hence $f_{\mathrm{G}^{-1}(A)}=0$ for every $A\in Fin$ with $1\notin A$. It follows that $f_A=0$ for every $A\in Fin$, consequently $F=0$. □

LEMMA 4.2. *J^* has trivial kernel.*

Proof. Let

$$F(x,\underline{i})=\sum_{A\in Fin}f_A(x)(-1)^{A(\underline{i})}.$$

Then

$$U^*_{I_n}\left(f_A\otimes(-1)^{A(\cdot)}\right)(x,\underline{i})=\begin{cases} f_A(S^{-n}x)(-1)^{\mathrm{G}^{-1}(A-n)(\underline{i})} & \text{if} \quad n+1\notin A\\ 0 & \text{if} \quad n+1\in A.\end{cases}$$

It follows that

$$
\begin{aligned}
J^*F(x,\underline{i}) &= \sum_{A\in Fin}\ \sum_{n\in\mathbb{Z},n+1\notin A} a_n f_A(S^{-n}x)(-1)^{\mathrm{G}^{-1}(A-n)(\underline{i})} \\
&= \sum_{B\in Fin}\sum_{n\in\mathbb{Z}} a_n f_{\mathrm{G}(B)+n}(S^{-n}x)(-1)^{B(\underline{i})} \\
&= \sum_{A\in Fin,1\notin A}\ \sum_{n\in\mathbb{Z}} a_n f_{A+n}(S^{-n}x)(-1)^{\mathrm{G}^{-1}(A)(\underline{i})}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
J^*F(x,\underline{i}) &= \sum_{A\in Fin_0}\ \sum_{k\in\mathbb{Z},1\notin A-k}\ \sum_{n\in\mathbb{Z}} a_n f_{A+n-k}(S^{-n}x)(-1)^{\mathrm{G}^{-1}(A-k)(\underline{i})} \\
&= \sum_{A\in Fin_0}\ \sum_{k\in\mathbb{Z},1\notin A-k} [\bar{a}*\left(\zeta^A(S^{-k}x)\right)]_k(-1)^{\mathrm{G}^{-1}(A-k)(\underline{i})},
\end{aligned}
$$

where $\zeta^A(x)=(\zeta^A(x)_l)_{l\in\mathbb{Z}}$ is given by $\zeta^A(x)_l=f_{A-l}(S^lx)$.

Suppose that $J^*(F)=0$. It follows that $[\bar{a}*\zeta^A(S^{-k}x)]_k=0$ for every $A\in Fin_0$, $k+1\notin A$ and for a.e. $x\in X$. Hence $\bar{a}*\left(\zeta^A(x)\right)\in l_0(\mathbb{Z})$ for μ-a.e. $x\in X$ (the only possibly non-zero terms of the convolved sequence have indices belonging to $A-1$). Since $\zeta^A(x)\in l^2(\mathbb{Z})$, in view of (3.1), $\zeta^A(x)=\overline{0}$ for every $A\in Fin_0$ and for μ-a.e. $x\in X$. Thus $f_A=0$ for all $A\in Fin$ and consequently $F=0$. □

It follows from the above two lemmas that the ranges of J and J^* are dense. Clearly J and J^* intertwine the Koopman operators U_{T_1} and U_{T_2}, hence we have proved the following.

Proposition 4.3. *Under the above notation the automorphisms T_1 and T_2 are Markov quasi-similar.* □

Recall that in [**13**] constructions of the above type have been used to produce weakly isomorphic transformations that are not isomorphic. In fact our transformation T_1 is the same as the transformation $T_{\dots,-1,0,1,2,\dots}$ in Subsection 4.2 in [**13**], where it is proved that each metric endomorphism that commutes with T_1 is invertible. It follows that T_1 cannot be a factor of the system given by its **proper** factor; in particular, it is not weakly isomorphic to T_2. In other words we have proved the following.

Proposition 4.4. *There are ergodic automorphisms which are Markov quasi-similar but they are not weakly isomorphic.* □

Remark 4.5. The Markov quasi-similarity between T_1 and T_2 constructed above is given by a $1-1$ Markov operator with dense range, that is, in fact we have shown that U_{T_1} and U_{T_2} are Markov quasi-affine. The Markov operator is given as a convex combination of isometries which separately have no dense ranges as they are not onto (and obviously their ranges are closed).

Let us emphasize that not each non-trivial choice of weights (a_n) gives rise to an operator with dense range as the following example shows.

EXAMPLE 4.6. Set $a_n = \frac{1}{2^{n+1}}$ for $n \geq 0$ and $a_n = 0$ for $n < 0$. We will show that in this case $\ker J^* \neq \{0\}$. Denoting by $\overline{S}$ the automorphism of $(X \times \mathbb{Z}_2^{\mathbb{Z}}, \mu \otimes \lambda_{\mathbb{Z}_2^{\mathbb{Z}}})$ given by

$$\overline{S}(x, \underline{i}) = (Sx, \ldots, i_{-1}, i_0, \overset{0}{i_1}, i_2, \ldots),$$

we have $I_n = I_0 \circ \overline{S}^n$ for any $n \in \mathbb{Z}$, and hence

$$J^* = U^*_{I_0} \circ \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} U_{\overline{S}^{-n}}.$$

In fact, we will prove that

$$\left(-\frac{1}{2}U_{\overline{S}^{-1}} + Id\right)\left(\ker U^*_{I_0}\right) \subset \ker J^*. \tag{4.2}$$

Notice that if $0 \neq G \in L^2(X \times \mathbb{Z}_2^{\mathbb{Z}}, \mu \otimes \lambda_{\mathbb{Z}_2}^{\mathbb{Z}})$ then $-\frac{1}{2}G \circ \overline{S}^{-1} + G \neq 0$ because the norms of the two summands are different. To prove (4.2) take $G \in \ker U^*_{I_0}$ and let $F = -\frac{1}{2}G \circ \overline{S}^{-1} + G$. Thus

$$\begin{aligned} J^*F &= U^*_{I_0}\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} F \circ \overline{S}^{-n}\right) \\ &= U^*_{I_0} \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} G \circ \overline{S}^{-n} - \frac{1}{2^{n+2}} G \circ \overline{S}^{-n-1}\right) = U^*_{I_0}\left(\frac{1}{2}G\right) = 0. \end{aligned}$$

Since $\ker U^*_{I_0}$ is not trivial, the claim follows.

5. *Metric invariants of Markov quasi-similarity*

By Proposition 2.1 the Markov quasi-similarity is stronger than spectral equivalence of Koopman representations (it will be clear from the results of this section that it is essentially stronger). In particular all spectral invariants like ergodicity, weak mixing, mild mixing, mixing and rigidity are invariants for Markov quasi-similarity. It also follows that each transformation which is spectrally determined, that is for which spectral equivalence is the same as measure-theoretical equivalence, is also Markov quasi-equivalence unique (up to measure-theoretic isomorphism). In particular each automorphism Markov quasi-similar to an ergodic transformation with discrete spectrum is isomorphic to it. The same holds for Gaussian-Kronecker systems (see [**5**]).

This spectral flavor is still persistent when we consider Markov quasi-images. Indeed, each Markov operator between L^2-spaces "preserves" the subspace of zero mean functions, therefore a direct consequence of (2.3) is that a transformation which is a Markov quasi-image of an ergodic (weakly mixing, mixing) system remains ergodic (weakly mixing, mixing). Despite all this, Markov quasi-similarity is far from being spectral equivalence. In order to justify this statement, we need a non-disjointness result from [**17**] (in fact its proof) which we now briefly recall.

Assume that T_i is an ergodic automorphism of $(X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, and let $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ be a Markov operator intertwining U_{T_1} and U_{T_2}. Then Φ sends L^∞-functions to L^∞-functions and we can consider H_Φ, the L^2-span of

$$\{\Phi(f_1^{(1)}) \cdot \ldots \cdot \Phi(f_m^{(1)}) : \; f_i^{(1)} \in L^\infty(X_1, \mathcal{B}_1, \mu_1), \; i = 1, \ldots, m, \; m \geq 1\}.$$

It turns out that $H_\Phi = L^2(\mathcal{A}_\Phi)$ where $\mathcal{A}_\Phi \subset \mathcal{B}_2$ is a T_2-invariant σ-algebra (in other words Φ defines a factor of T_2). Then by the proof of the main non-disjointness result (Theorem 4) in [**17**] this factor is also a factor of an (ergodic) infinite self-joining of T_1. If we assume additionally that $\operatorname{Im}\Phi$ is dense then $H_\Phi = L^2(X_2, \mathcal{B}_2, \mu_2)$ and the factor given by $\mathcal{A}_\Phi$ is equal to T_2 itself.

PROPOSITION 5.1. *If T_2 is a Markov quasi-image of T_1 then T_2 is a factor of some infinite ergodic self-joinings of T_1.* □

As all the systems determined by (infinite) joinings of zero entropy systems have zero entropy and the systems given by joinings of distal systems are also distal (for these results see e.g. [**7**]), Proposition 5.1 yields the following conclusion.

PROPOSITION 5.2. *Each automorphism which is a Markov quasi-image of a zero entropy system has zero entropy. Each automorphism which is a Markov quasi-image of a distal system remains distal. In particular, zero entropy and distality are invariants of Markov quasi-similarity in the class of measure-preserving systems.* □

As a matter of fact, we can prove that zero entropy is an invariant of Markov quasi-similarity in the class of measure-preserving systems in a simpler manner. Recall that T_1 and T_2 are said to be *disjoint* (in the sense of Furstenberg [**6**]) if the only joining between them is the product measure. The following result will help us to indicate further invariants of Markov quasi-similarity.

LEMMA 5.3. *If T_1 is disjoint from S and T_2 is a Markov quasi-image of T_1 then T_2 is also disjoint from S.*

Proof. Indeed, assume that $\Phi \circ U_{T_1} = U_{T_2} \circ \Phi$ and Φ has dense range. If T_2 and S are not disjoint then we have a non-trivial Markov operator Ψ intertwining U_{T_2} and U_S. Since Φ has dense range, $\Psi \circ \Phi$ is a non-trivial Markov operator intertwining U_{T_1} and U_S and therefore T_1 is not disjoint from S. □

Given a class $\mathcal{M}$ of automorphisms denote by $\mathcal{M}^\perp$ the class of those transformations which are disjoint from all members of $\mathcal{M}$. In view of Lemma 5.3 we have the following.

PROPOSITION 5.4. *$\mathcal{M}^\perp$ is closed under taking automorphisms which are Markov quasi-images of members of $\mathcal{M}^\perp$. In particular, if $\mathcal{M} = \mathcal{M}^{\perp\perp}$ then $\mathcal{M}$ is closed under taking automorphisms which are Markov quasi-images of members of $\mathcal{M}$.* □

If by $\mathcal{K}$ and $\mathcal{ZE}$ we denote the classes of Kolmogorov automorphisms and zero entropy automorphisms respectively then we have $\mathcal{K} = \mathcal{ZE}^\perp$ ([**6**]) and therefore by Proposition 5.4 we obtain the following.

COROLLARY 5.5. *Every automorphism which is a Markov quasi-image of a Kolmogorov automorphism is also K. In particular, K property is an invariant of Markov quasi-similarity in the class of measure-preserving systems.* □

PROBLEM 1. Is the same true for Bernoulli automorphisms?

Notice that also $\mathcal{ZE} = \mathcal{K}^\perp$. Therefore we can apply Proposition 5.4 with $\mathcal{M} = \mathcal{ZE}$ to obtain that an automorphism which is a Markov quasi-image of a zero entropy system has zero entropy.

6. *JP property and Markov quasi-similarity*

DEFINITION 1. An ergodic automorphism T on $(X, \mathcal{B}, \mu)$ is said to have *the joining primeness* (JP) property (see [**16**]) if for each pair of weakly mixing automorphisms S_1 on $(Y_1, \mathcal{C}_1, \nu_1)$ and S_2 on $(Y_2, \mathcal{C}_2, \nu_2)$ and for every indecomposable Markov operator

$$\Phi : L^2(X, \mu) \to L^2(Y_1 \times Y_2, \nu_1 \otimes \nu_2)$$

intertwining U_T and $U_{S_1 \times S_2}$ we have (up to some abuse of notation) $\operatorname{Im} \Phi \subset L^2(Y_1, \mathcal{C}_1, \nu_1)$ or $\operatorname{Im} \Phi \subset L^2(Y_2, \mathcal{C}_2, \nu_2)$.

The class of JP automorphisms includes in particular the class of simple systems ([**10**]). For other natural classes of JP automorphisms including some smooth systems see [**16**] (we should however emphasize that a "typical" automorphism is JP [**16**]).

Assume that T is JP and $S_1, S_2, \ldots$ are weakly mixing. Let $\Phi : L^2(X, \mu) \to L^2(Y_1 \times Y_2 \times \ldots, \nu_1 \otimes \nu_2 \otimes \ldots)$ be a Markov operator intertwining U_T and $U_{S_1 \times S_2 \times \ldots}$. Let $\Phi = \int_\Gamma \Phi_\gamma \, dP(\gamma)$ be the decomposition corresponding to the ergodic decomposition of the joining determined by Φ. Slightly abusing notation, we claim that for P-a.e. $\gamma \in \Gamma$

$$\Phi_\gamma(L^2(X, \mathcal{B}, \mu)) \subset L^2(Y_{i_\gamma}, \mathcal{C}_{i_\gamma}, \nu_{i_\gamma}), \text{ for some } i_\gamma \in \{1, 2, \ldots\}.$$

Indeed, we use repeatedly the definition of JP property: We represent $\Pi_{n \geq 1} S_n$ as $S_1 \times (\Pi_{n \geq 2} S_n)$ and if $\operatorname{Im} \Phi_\gamma$ is not included in $L^2(Y_1, \nu_1)$ then $\operatorname{Im} \Phi_\gamma \subset L^2(Y_2 \times Y_3 \times \ldots, \nu_2 \otimes \nu_3 \otimes \ldots)$. In the next step we write $\Pi_{n \geq 1} S_n = (S_1 \times S_2) \times (\Pi_{n \geq 3} S_n)$ and we check if $\operatorname{Im} \Phi_\gamma \subset L^2(Y_1 \times Y_2, \nu_1 \otimes \nu_2)$ (if it is the case then $\operatorname{Im} \Phi_\gamma \subset L^2(Y_2, \nu_2)$); if it is not the case then $\operatorname{Im} \Phi_\gamma \subset L^2(Y_3 \times Y_4 \times \ldots, \nu_3 \otimes \nu_4 \otimes \ldots)$, etc. If for each $n \geq 1$, $\operatorname{Im} \Phi_\gamma \perp L^2(Y_1 \times \ldots \times Y_n, \nu_1 \otimes \ldots \otimes \nu_n)$, then $\operatorname{Im} \Phi_\gamma = 0$ (since functions depending on finitely many coordinates are dense), and hence $\Phi_\gamma = 0$.

It follows that for some $0 \leq a_n \leq 1$ with $\sum_{n \geq 1} a_n = 1$

$$\Phi = \sum_{n \geq 1} a_n \Phi_n, \tag{6.1}$$

where $\operatorname{Im} \Phi_n \subset L^2(Y_n, \mathcal{C}_n, \nu_n)$. In particular,

$$\operatorname{Im} \Phi \subset \bigoplus_{n \geq 1} L^2(Y_n, \mathcal{C}_n, \nu_n) \subset L^2(Y_1 \times Y_2 \times \ldots, \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \ldots, \nu_1 \otimes \nu_2 \otimes \ldots). \tag{6.2}$$

Note that the space $F := \bigoplus L^2(Y_n, \nu_n)$ is closed and $U_{S_1 \times S_2 \times \ldots}$-invariant.

LEMMA 6.1. *Under the above notation, if $\mathcal{A} \subset \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \ldots$ is a factor of $S_1 \times S_2 \times \ldots$ and it is also a Markov quasi-image of a JP automorphism T then there exists $n_0 \geq 1$ such that $\mathcal{A} \subset \mathcal{C}_{n_0}$; in other words the factor given by $\mathcal{A}$ is a factor of S_{n_0}.*

Proof. Asume that Φ intertwines U_T and the Koopman operator of the factor action of $S_1 \times S_2 \times \ldots$ on $\mathcal{A}$. Since the range of Φ is dense in $L^2(\mathcal{A})$, it follows that $\Phi : L^2(X, \mathcal{B}, \mu) \to L^2(\mathcal{A}) \subset F$. We now use an argument from [**9**]: Take $A \in \mathcal{A}$. In view of (6.2) we have

$$\mathbf{1}_A - (\nu_1 \otimes \nu_2 \otimes \ldots)(A) = f_1(y_1) + f_2(y_2) + \ldots$$

with $f_n \in L^2_0(Y_n, \nu_n)$, $n \geq 1$. Since the distribution of the random variable $\mathbf{1}_A - (\nu_1 \otimes \nu_2 \otimes \ldots)(A)$ is a measure on a two element set and the random variables $f_1, f_2, \ldots$ are independent, all of them but one, say f_{n_A}, are equal to zero. In other words, $A \in \mathcal{C}_{n_A}$. It easily follows that the function $\mathcal{A} \ni A \mapsto n_A$ is constant (see [**9**]). □

Let T be a simple weakly mixing automorphism. By the definition of simplicity, it follows that each of its ergodic infinite self-joinings is, as a dynamical system, isomorphic to a Cartesian product $T^{\times n}$ with $n \leq \infty$. Since each simple system has the JP property, in view of Proposition 5.1 and Lemma 6.1 (in which $S_n = T$) we obtain the following.

PROPOSITION 6.2. *Each automorphism which is a Markov quasi-image of a simple map T is a factor of T.* □

It follows from the above proposition that if T_1 and T_2 are weakly mixing simple automorphisms and are Markov quasi-similar then they are isomorphic.

REMARK 6.3. In our example of T_1 and T_2 non-weakly isomorphic but Markov quasi-similar T_2 is a factor of T_1 but (because of absence of weak isomorphism) T_1 is not a factor of T_2. Hence the family of factors of T_2 is strictly included in the family of automorphisms which are Markov quasi-images of T_2.

When we apply Proposition 6.2 to the MSJ maps (see [**10**]) we obtain that such systems are Markov quasi-similarly prime, that is we have the following.

COROLLARY 6.4. *The only non-trivial automorphism which is a Markov quasi-image of an MSJ system T is T itself.* □

REMARK 6.5. Assume that T enjoys the MSJ property. Take Φ_1, Φ_2 two joinings of T and $T \times T$ so that $\operatorname{Im}\Phi_1 \cap \left(L^2(X,\mu) \otimes \mathbf{1}_X\right) \neq \{0\}$ and $\operatorname{Im}\Phi_2 \cap \left(\mathbf{1}_X \otimes L^2(X,\mu)\right) \neq \{0\}$. Then $\Phi := a\Phi_1 + (1-a)\Phi_2$ is a Markov operator intertwining U_T and $U_{T\times T}$ and if $0 < a < 1$, then the range of Φ is **not** dense in $L^2(\mathcal{A}_\Phi)$. Indeed, $\mathcal{A}_\Phi$ is either $T \times T$ or $T \odot T$ (the factor of $T \times T$ determined by the σ-algebra of sets invariant under exchange of coordinates) and the claim follows from Lemma 6.1. This is the answer to a question raised by François Parreau in a conversation with the second named author of the note.

It means that if we try to define Markov quasi-image by requiring that $\mathcal{A}_\Phi = \mathcal{B}_2$ instead of requiring that the range of Φ is dense in $L^2(X_2, \mathcal{B}_2, \mu_2)$ then we obtain a strictly weaker notion.

7. *Final remarks and problems*

Notice that the joining of T_1 and T_2 corresponding to the Markov operator in Section 4 and based on constructions from [**13**] is not ergodic (i.e. the Markov operator is decomposable). In fact, in our construction of two non-weakly isomorphic Markov quasi-similar automorphisms T_1 and T_2 no Markov operator corresponding to an ergodic joining between T_1 and T_2 can have dense range. Indeed, first recall that ergodic Markov quasi-similar automorphisms have the same Kronecker factors. Then notice that T_1 and T_2 are compact abelian group extensions of the same (in [**13**] this is the classical adding machine system) Kronecker factor. Hence, assume that T is an ergodic automorphism with discrete spectrum and let $\phi : X \to G$, $\psi : X \to H$ be ergodic cocycles with values in compact abelian groups G and H respectively. We then have the following.

$$\begin{array}{c}T_\phi \text{ and } T_\psi \text{ are Markov quasi-similar via } \textbf{indecomposable} \\ \text{Markov operators if and only if they are weakly isomorphic.}\end{array} \tag{7.1}$$

Indeed, every ergodic joining between such systems is the relatively independent extension of the graph joining given by an isomorphism I of so called natural factors $T_{\phi J}$ and $T_{\psi F}$ acting on $X \times G/J$ and $X \times H/F$ respectively, see [**14**]. The Markov operator Φ corresponding to such a joining is determined by the orthogonal projection on the $L^2(X \times H/F, \mu \otimes \lambda_{H/F})$; in particular the range of Φ is closed. Therefore it has dense range only if $\mathrm{Im}\,\Phi = L^2(X \times H, \mu \otimes \lambda_H)$ which means that in fact I settles a metric isomorphism of T_ψ and a factor of T_ϕ. In other words, T_ψ is a factor of T_ϕ.

This shows that there exist two ergodic automorphisms which are Markov quasi-similar but Markov quasi-similarity cannot be realized by indecomposable Markov operators with dense ranges.

We have been unable to construct an indecomposable $1-1$ Markov operator Φ with dense range intertwining the Koopman operators given by two non-isomorphic ergodic automorphisms T_1 and T_2. One might think about such a construction using Markov operators given as convex combinations of U_{S_i} where S_i are space isomorphisms which are **not** intertwining T_1 and T_2 (see e.g. [**2**] for the notion of near simplicity where a similar idea is applied).

It seems that Proposition 2.1 rules out a possibility to find two Markov quasi-similar Gaussian automorphisms which are not isomorphic by a use of so called Gaussian joinings [**17**] (recall that Gaussian joinings are ergodic joinings). Indeed, once a Markov quasi-similarity is given by an integral of Markov operators corresponding to Gaussian joinings, it sends chaos into chaos (see [**17**] for details). In particular, first chaos is sent into first chaos, and we obtain a quasi-similarity of the unitary actions restricted to the first chaos. By Proposition 2.1 these actions on the first chaos are spectrally equivalent which in turn implies measure-theoretic isomorphism of the Gaussian systems.

We do not know however if we can have two non-weakly isomorphic Poisson suspension systems which are Markov quasi-similar by a use of Poissonian joinings (which are ergodic), see [**3**] and [**22**].

PROBLEM 2. Recall that in the construction carried out in Section 4, T_2 was a factor of T_1. Is it possible to construct Markov quasi-similar automorphisms T_1 and T_2 such that T_1 and T_2 have no common (non-trivial) factors?

Of course such T_1 and T_2 must not be disjoint (see [**6**]). The most “popular” construction of a pair of non-disjoint systems without common factors is $(T, T \odot T)$ (for a particular T; see [**9**], [**23**]). Notice however that these two automorphisms are not Markov quasi-similar if T has the JP property (see Lemma 6.1), that is, in all known cases where T and $T \odot T$ have no common (isomorphic) non-trivial factors.

PROBLEM 3. As we have already noticed in Remark 2.2, Markov quasi-affinity implies Markov quasi-similarity. Are these notions equivalent? If the answer is positive then each weakly isomorphic transformations would have to be Markov quasi-affine. Are examples of weakly isomorphic non-isomorphic automorphisms from [**12**], [**13**] or [**23**] Markov quasi-affine?

PROBLEM 4. The examples of Markov quasi-similar automorphisms which are not isomorphic presented in this note have infinite spectral multiplicity. Is it possible to find such examples in the class of systems with simple spectrum (or of finite spectral multiplicity)? In the class of rank one systems?

Recall that in case of finite spectral multiplicity systems their weak isomorphism implies isomorphism, see e.g. [**18**].

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References

1. H. Anzai, 'Ergodic skew product transformations on the torus', *Osaka J. Math.* 3 (1951), 83–99.
2. A. Danilenko, 'On simplicity concepts for ergodic actions', *J. Anal. Math.* 102 (2007), 77–117.
3. Y. Derriennic, K. Frączek, M. Lemańczyk and F. Parreau, 'Ergodic automorphisms whose weak closure of off-diagonal measures consists of ergodic self-joinings', *Coll. Math.* 110 (2008), 81-115.
4. C. Foiaş and B. Sz.-Nagy, *Harmonic analysis of operators on Hilbert space*, (North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest 1970).
5. C. Foiaş and S. Stratila, 'Ensembles de Kronecker dans la théorie ergodique', *C.R. Acad. Sci. Paris, sr. A-B* 267 (1968), A166–A168.
6. H. Furstenberg, 'Disjointness in ergodic theory, minimal sets, and a problem of Diophantine approximation', *Math. Systems Th.* 1 (1967), 1–49.
7. E. Glasner, *Ergodic Theory via Joinings*, (Math. Surveys Monogr. 101, Amer. Math. Soc., Providence, RI, 2003).
8. C. Hoffman, 'A K counterexample machine', *Trans. Amer. Math. Soc.* 351 (1999), 4263–4280.
9. A. del Junco and M. Lemańczyk, 'Generic spectral properties of measure-preserving maps and applications', *Proc. Amer. Math. Soc.* 115 (1992), 725–736.
10. A. del Junco and D. Rudolph, 'On ergodic actions whose self-joinings are graphs', *Ergodic Theory Dynam. Systems* 7 (1987), 531–557.
11. A.B. Katok and J.-P. Thouvenot, 'Spectral Properties and Combinatorial Constructions in Ergodic Theory', *Handbook of dynamical systems. Vol. 1B* (Elsevier B. V., Amsterdam, 2006) 649–743.
12. J. Kwiatkowski, M. Lemańczyk and D. Rudolph, 'Weak isomorphism of measure-preserving diffeomorphisms', *Israel J. Math.* 80 (1992), 33–64.
13. M. Lemańczyk, 'Weakly isomorphic transformations that are not isomorphic', *Probab. Theory Related Fields* 78 (1988), 491–507.
14. M. Lemańczyk and M.K. Mentzen, 'Compact subgroups in the centralizer of natural factors of an ergodic group extension of a rotation determine all factors', *Ergodic Theory Dynam. Systems* 10 (1990), 763–776.
15. M. Lemańczyk and F. Parreau, 'Lifting mixing properties by Rokhlin cocycles', Preprint 2004.
16. M. Lemańczyk, F. Parreau and E. Roy, 'Joining primeness and disjointness from infinitely divisible systems', Preprint 2004.
17. M. Lemańczyk, F. Parreau and J.-P. Thouvenot, 'Gaussian automorphisms whose ergodic self-joinings are Gaussian', *Fund. Math.* 164 (2000), 253–293.
18. D. Newton, 'Coalescence and spectrum of automorphisms of a Lebesgue space', *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 19 (1971), 117–122.
19. D.S. Ornstein, *Ergodic theory, randomness, and dynamical systems*, James K. Whittemore Lectures in Mathematics given at Yale University. Yale Mathematical Monographs, No. 5. (Yale University Press, New Haven, Conn.-London, 1974).
20. W. Parry, *Topics in ergodic theory*, Cambridge Tracts in Mathematics, 75. (Cambridge University Press, Cambridge-New York, 1981).
21. S. Polit, *Weakly isomorphic maps need not be isomorphic*, Ph.D. Thesis, Stanford University, 1974.
22. E. Roy, *Mesures de Poisson, infinie divisibilité et propriétés ergodiques*, Ph.D. Thesis, Université Paris 6, 2005.
23. D. Rudolph, 'An example of a measure preserving map with minimal self-joinings, and applications', *J. Analyse Math.* 35 (1979), 97–122.
24. V.V. Ryzhikov, 'Joinings, intertwining operators, factors and mixing properties of dynamical systems', *Russian Acad. Izv. Math.* **42** (1994), 91–114.
25. Ya.G. Sinai, 'On weak isomorphism of measure-preserving transformations', (Russian) *Mat. Sb.* 63 (1964), 23–42, *Transl. Amer. Math. Soc.* 57 (1966), 123–143 (English).
26. A.M. Vershik, 'Polymorphisms, Markov processes, and quasi-similarity', *Discrete and Continuous Dynam. Systems* 13 (2005), 1305–1324.

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