Wild hereditary algebras

based on the talk by Otto Kerner (Düsseldorf)

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In our talk Q be a finite quiver without oriented cycles. Denote by $Q_0 = \{1, 2, \ldots, n\}$ the set of vertices, and by Q_1 the set of arrows. For an arrow α we denote by $s(\alpha)$ the source of α and by $t(\alpha)$ the target. The quiver Q is wild if and only if Q is neither Dynkin nor Euclidean. We assume that Q is a wild quiver. During the talk k be any field and $H = kQ$. Then $\operatorname{mod} H$ is equivalent to $\operatorname{rep}_k Q$.

For a hereditary algebra $\tau := D\operatorname{Tr} = D\operatorname{Ext}(-, H)$ is left exact and $\tau^- := \operatorname{Tr} D = \operatorname{Ext}(D-, H)$ is right exact. We have the Nakayama functors $\nu := D\operatorname{Hom}(-, H)$ and $\nu^- = \operatorname{Hom}(D-, H)$, and from any short exact sequence $0 \to U \to V \to W \to 0$ we get an exact sequence $0 \to \tau U \to \tau V \to \tau W \to \nu U \to \nu V \to \nu W \to 0$ and the dual exact sequence involving τ^- and ν^-.

For each module $V \in \operatorname{mod} H$ we can define $\mathbf{dim}\, V \in \mathbb{Z}^n$. We have the Coxeter transformation $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$ given by $\Phi(\mathbf{dim}\, P_i) = -\mathbf{dim}\, I_i$, which is invertible over $\mathbb{Z}$. It follows that if X has not projective direct summand then $\mathbf{dim}\, \tau X = \Phi(\mathbf{dim}\, X)$.

The Euler form $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ is given by $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^n x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{e(\alpha)}$. It can be proved that $\langle \mathbf{dim}\, X, \mathbf{dim}\, Y \rangle = \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}^1(X, Y)$.

We can divide $\operatorname{ind} H$ into the preprojective component $\mathcal{P}(H)$, the preinjective component $\mathcal{I}(H)$ and the regular part $\Omega(H)$.

We have the following properties of preprojective modules.

(a) If X is an indecomposable preprojective module then $\operatorname{End}(X) = K$ and $\operatorname{Ext}(X, X) = 0$, hence $\langle \mathbf{dim}\, X, \mathbf{dim}\, X \rangle = 1$.

(b) If X is an indecomposable preprojective module and Y is any indecomposable module with $\mathbf{dim}\, Y = \mathbf{dim}\, X$, then $X \simeq Y$.

(c) For each natural number N there exists a natural number s such that each $\dim_K \tau^{-s} P \geq N$ for $s \gg 0$, where P is an indecomposable projective module.

We have also stronger versions of the last result.

(1) $\limsup \dim \operatorname{Hom}(P_i, \tau^{-s}P) = \infty$ for each i.
(2) $\operatorname{Hom}(P_i, \tau^{-s}P) \neq 0$ for $s \gg 0$, hence almost all modules in $\mathcal{P}(H)$ are sincere
(3) $\lim \dim \operatorname{Hom}(P_i, \tau^{-s}P) = \infty$.
(4) $\lim \sqrt[s]{\dim \operatorname{Hom}(P_i, \tau^{-s}P)} = \rho > 1$ where ρ is the spectral radius of Φ and is also called the growth number.
(5) $\lim \frac{\dim \operatorname{Hom}(P_i, \tau^{-s}P)}{\rho^s} > 0$.

The first two results were proved by Ringel in 1988 and the last one follows from result of Ringel from 1993.

We call a module R regular if $\tau^{-m}\tau^m R \simeq R$ for all $m \in \mathbb{Z}$. By $\operatorname{reg} H$ we will denote the full subcategory of $\operatorname{mod} H$ of regular modules. The category $\operatorname{reg} H$ is closed under extensions, images, but is not closed under kernels and cokernels, hence is not abelian.

Lemma. *If $X \neq 0$ is regular then there exists N, depending only on* $\dim X$, *with the property that for each regular H-module X' with* $\dim X' \leq \dim X$ *and for each homomorphism $f : \tau^s X' \to R$, with R regular and $s \geq N$,* $\operatorname{Ker} f$ *is regular.*

Proof. Let $K := \operatorname{Ker} f$. Obviously K has no preinjective direct summand, hence we have $K = K_p \oplus K_r$, where K_p is preprojective and K_r is regular. The short exact sequence $0 \to K \to \tau^s X' \to R \to 0$ remains exact after applying τ^{-s}. But $\tau^{-s}\tau^s X' = X'$ and $\dim \tau^{-s} K_p > \dim X$ for $s \gg 0$, if $K_p \neq 0$, and we get a contradiction. $\square$

Of course, we have also a dual statement for cokernels.

Assume $R \simeq \tau^m R$ for $m > 0$ for some regular module R. Then for $\mathbf{x} = \mathbf{dim}\, R + \cdots + \mathbf{dim}^{m-1} R$ we have $\Phi(\mathbf{x}) = \mathbf{x}$. However, it is impossible for a wild quiver. Hence $R \not\simeq \tau^m R$ for any $m \in \mathbb{Z}$. Even $\mathbf{dim}\, R \neq \mathbf{dim}\, \tau^m R$ for any m.

Lemma. *Let $f : R \to \tau^{-i} R$ be a nonzero map for $i > 0$. Then f is neither injective nor surjective.*

Proof. If f is injective, then we have an infinite sequence of proper monomorphism

$$\cdots \to \tau^{2i} R \xrightarrow{\tau^{2i} f} \tau^i R \xrightarrow{\tau^i f} R \xrightarrow{f} \tau^{-i} R,$$

and this is a contradiction. $\square$

Lemma. *We have the following.*

(a) *Let $R \neq 0$ be a regular module without proper regular factors. Then there exist a natural number $m > 0$ and an exact sequence $0 \to R \to \tau^m R \to Q \to 0$ with Q preinjective.*

(b) *Let $R \neq 0$ be a regular module without proper regular submodules. Then there exist a natural number $m > 0$ and a short exact sequence $0 \to P \to \tau^{-m} R \to R \to 0$ with P preprojective.*

Proof. (a) Let $Y = R \oplus \tau^2 R \oplus \cdots \oplus \tau^{2n} R$. Then $D\operatorname{Hom}(Y, \tau Y) = \operatorname{Ext}(Y, Y) \neq 0$. In particular, we have there exist i and j such that $\operatorname{Hom}(R, \tau^{2(j-i)+1} R) \simeq \operatorname{Hom}(\tau^{2i} R, \tau^{2j+1} R) \neq 0$. Hence we have a nonzero morphism $f : R \to \tau^m R$. It follows that f must be injective, hence $m > 0$, and moreover the cokernel of f is preinjecitive, $\tau^m R$ has no proper regular factor.

The proof of (b) is dual. $\square$

If $X' \neq 0$ is regular then for some $t \geq 0$ $X := \tau^t X'$ have the property that for any homomorphism $f : X \to R$ we have that $\operatorname{Ker} f$ is regular. Hence we can choose an epimorphism $f : X \to R$ with the property that R has no proper regular factors and $\operatorname{Ker} f$ is regular. Then for each $s > 0$ we have an epimorphism $tau^s X \to \tau^s R$. Thus we get the following consequence of the above lemma and the properties of preinjective and preprojective modules.

Corollary. *Let $X \neq 0$ be a regular module.*

(a) $\dim \tau^m X \gg 0$ *for* $|m| \gg 0$.

(b) $\tau^m X$ *is sincere for* $|m| \gg 0$.

Main Lemma. *Let X and Y be nonzero regular modules. Then we have the following.*

(a) $\operatorname{Hom}(\tau^m X, Y) = 0$ *for* $m \gg 0$.

(b) $\operatorname{Hom}(X, \tau^m Y) \neq 0$ *for* $m \gg 0$.

The first point of the above lemma has been proved by Otto Kerner, and the second by D. Bear. We present here the proof of Frank Lükas.

Proof. (a) There exist only finitely many dimension vectors $\mathbf{x}_1, \ldots, \mathbf{x}_t$ of regular modules $R_1, \ldots, R_t$ with $\mathbf{x}_i \leq \mathbf{dim}\, Y$. There exists a natural number $N > 0$ such that $|\Phi^{-m}(\mathbf{x}_i)| = \dim \tau^{-m} R_i > \dim X$ for all i and $m \geq N$. Take $m \geq N$ and suppose $f : \tau^m X \to Y$ is a nonzero homomorphism. Let $R := \operatorname{Im} f$. Then R is a regular module and $\mathbf{dim}\, R \leq \mathbf{dim}\, Y$ hence $\mathbf{dim}\, R = \mathbf{x}_i$ for some i. Let $K := \operatorname{Ker} f$. Then we have an exact sequence $0 \to \tau^{-m} K \to X \to \tau^{-m} R \to 0$ and this gives a contradiction, since $\dim \tau^{-m} R = |\Phi^{-m}(\mathbf{x}_i)| > \dim X$.

(b) Assume X has no proper regular factors. We have an exact sequence $0 \to X \to \tau^s X \to Q \to 0$ with s positive and Q preinjective. There exists some $N > 0$ such that $\mathrm{Hom}(\tau^m Y, \tau^s X) = 0$ and $\mathrm{Hom}(\tau^m Y, Q) \neq 0$ for all $m \geq N$. Take $m > N$. We apply the functor $\mathrm{Hom}(-, \tau^m Y)$ to the above exact sequence and we get an exact sequence $\mathrm{Hom}(X, \tau^m Y) \to \mathrm{Ext}(Q, \tau^m Y) \longrightarrow (\tau^s X, \tau^m Y)$. However we have $\mathrm{Ext}(\tau^s X, \tau^m Y) \simeq D\,\mathrm{Hom}(\tau^{m-1} Y, \tau^s X) = 0$ and $\mathrm{Ext}(Q, \tau^m Y) \simeq \mathrm{Hom}(\tau^{m-1} Y, Q) \neq 0$. This implies $\mathrm{Hom}(X, \tau^m Y)$.

If X is general then X has a factor module without any proper regular factor, and we use the first part. □

Remark. In fact we have shown.

(a) Let X be a nonzero regular module and N some natural number. Then there exists m_0 such that $\mathrm{Hom}(\tau^m X, Y) = 0$ for all $m \geq m_0$ and any regular module Y with the property $\dim Y \leq N$.

(b) Let Y be a nonzero regular module and N some natural number. Then there exists m_1 such that $\mathrm{Hom}(X, \tau^m Y) \neq 0$ for all $m \geq m_1$ and any regular module X with the property $\dim X \leq N$.

We have the following corollary.

Corollary. *Let X and Y be regular nonzero modules. Then we have following.*

(a) *X is cogenerated by $\tau^m Y$ for $m \gg 0$.*

(b) *Y is generated by $\tau^{-m} X$ for $m \gg 0$.*