Elementary modules

based on the talk by Otto Kerner (Düsselford)

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In our talk k be any field. Let H be a connected hereditary wild algebra. A regular H-module E is called **elementary** if there is not short exact sequence $0 \to U \to E \to V \to 0$ where U and V are nonzero regular H-modules.

Lemma. (a) *If E is an elementary module then $\tau^l E$ is elementary for all $l \in \mathbb{Z}$.*

(b) *If E is an elementary module then E is quasi–simple and* $\operatorname{End}(E)$ *is a division algebra.*

(c) *For E regular the following are equivalent.*

  (i) *E is elementary.*

  (ii) *$\tau^l E$ has no proper regular factors for $l \gg 0$.*

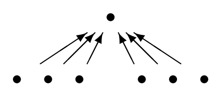
  (iii) *$\tau^{-l} E$ has no proper regular submodules for $l \gg 0$.*

Theorem (Lükas)**.** *Let H be connected wild hereditary algebra. Then the set $\{(\mathbf{dim}\, \tau^i E)_{i \in \mathbb{Z}} \mid E$ elementary$\}$ is finite.*

Proof. The proof is based on the following observations. There exists a natural number N such that if E is an elementary module then $\dim \tau^i E \leq N$ for some i. □

Let k be algebraically closed and H be the path algebra of following quiver $\cdot \leftleftarrows \cdot \leftarrow \cdot$. We can use the idea of the proof to calculate elementary modules. Namely, we have that an indecomposable H-module E is elementary if and only if $\mathbf{dim}\, \tau^i E = (1, 1, 0)$ or $\mathbf{dim}\, \tau^i E = (1, 2, 0)$ for some i.

An indecomposable regular H-module E is called **additively elementary** if each short exact sequence $0 \to U \to E^r \to V \to 0$, where U and V are regular, splits. We know there exist elementary modules which are not additively elementary. Consider the path algebra of quiver



and let E be a quasi–simple $k\widetilde{\mathbb{D}}_4$-module with the dimension-vector $\begin{smallmatrix} & & & 2 & & & \\ 1 & 1 & 1 & & 1 & 0 & 0 \end{smallmatrix}$. Then E is an elementary H-module. However, we have an exact sequence $0 \to X \to E^2 \to Y \to 0$, with X and Y regular H-modules of dimension vectors $\begin{smallmatrix} & & & 3 & & & \\ 1 & 1 & 1 & & 1 & 0 & 0 \end{smallmatrix}$ and $\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & 1 & & 1 & 0 & 0 \end{smallmatrix}$ respectively.

Theorem. *Let E be a quasi–simple regular module with* $\operatorname{Ext}(E, E) = 0$. *Then the following are equivalent.*

(a) *E is elementary.*

(b) *E is additively elementary.*

(c) *There exists a natural number m_0 such that for any regular module R the minimal right approximation $f : \tau^l E^s \to R$, with $l \geq m_0$, is a monomorphism.*

(d) *There exists a natural number m_0 such that for each $l \geq m_0$ $\tau^l E \oplus M$ is a tilting module for some preinjective module M.*

Proof. (b) $\Rightarrow$ (a) is obvious.

(d) $\Rightarrow$ (b). Denote $\tau^l E$ by E' and consider a short exact sequence $0 \to U \to (E')^r \to V \to 0$ with U and V regular. We apply the functor $\operatorname{Hom}(M, -)$ and we get $0 \to \operatorname{Ext}(M, U) \to 0 \to \operatorname{Ext}(M, V) \to 0$. Hence U and V belongs to $M^\perp = \operatorname{add} E$ and the sequence splits.

(a) $\Rightarrow$ (d). Take m_0 such that $\tau^l E$ has no regular factors and is sincere for $l \geq m_0$. Let $l \geq m_0$ and $E' := \tau^l E$. Then E' is faithful, since it is sincere without selfextensions. Let M be a cokernel of a monomorphism $H \to (E')^r$. Then $T := E' \oplus M$ is a titling H-module. Note that the torsion class $\mathcal{T}(T)$ is generated by E'.

We have to show that M is preinjective. Let V be an indecomposable direct summand of M. Assume V is not preinjective. Then V is regular. We have a nonzero map $f : E' \to V$, which has to be a monomorphism, since E' has no regular factors. Hence it follows that $\dim \operatorname{Hom}(E', V) > 1$. Let $Q := \operatorname{Coker} f$. Then $Q \in \mathcal{T}$. We use the following lemma.

Lemma (Unger)**.** *Let X and Y be nonisomorphic indecomposable modules without selfextensions such that* $\operatorname{Hom}(X, Y) \neq 0$ *and* $\operatorname{Ext}(Y, X) = 0$. *Then either we have a monomorphism $f : X \to Y$ or an epimorphism $g : Y \to X$ such that for $Q :=$* $\operatorname{Coker} f$ *(respectively $Q :=$* $\operatorname{Ker} g$*) we have* $\operatorname{End}(Q) = K$ *and* $\dim \operatorname{Ext}(Q, Q) = \dim \operatorname{Hom}(X, Y) - 1$.

According to the above lemma we may assume that $\dim \operatorname{Ext}(Q, Q) > 0$, hence Q is regular. Let $\tau_{\mathcal{T}} := t_T \tau_H$ be the relative Auslander–Reiten translation, where t_T denotes the biggest torsion submodule of a given module. We have a short exact sequence $0 \to \tau E' \to \tau V \to \tau Q \to 0$. Note that $\operatorname{Hom}(E', \tau V) = \operatorname{Ext}(V, E') = 0$, hence when we apply the functor

$\operatorname{Hom}(E', -)$ we get an exact sequence $0 \to \operatorname{Hom}(E', \tau Q) \to \operatorname{Ext}(E', \tau E') \to \operatorname{Ext}(E', \tau V) = 0$, thus $\dim \operatorname{Hom}(E', \tau Q) = 1$. Then we have a monomorphism $E' \to \tau Q$, hence $\tau_T Q \simeq E'$. If $A := \operatorname{End}(T)$ then an indecomposable A-projective module $\operatorname{Hom}(T, E')$ have the property $\tau_A^- \operatorname{Hom}(T, E') = \operatorname{Hom}(T, Q)$ has selfextensions. However, we have maps from $\operatorname{Hom}(T, E')$ to all projective A-modules hence $\operatorname{Hom}(T, E')$ has to be preprojective, and this is a contradiction. $\square$

An indecomposable regular H-module E is called **orbital elementary** if for each $\widetilde{E}$ in $\operatorname{add}(\tau^i E \mid i \in \mathbb{Z})$ any exact sequence $0 \to U \to \widetilde{E} \to V \to 0$, with U and V regular, splits.

Additively elementary modules does not have to be orbital elementary. Let H be path algebra of the quiver $\cdot \leftleftarrows \cdot \leftarrow \cdot$ and let U_1 be a regular elementary H-module of dimension vector $(1, 2, 0)$. Then for $U_2 := \tau U_1$ we have $\mathbf{dim}\, U_2 = (3, 4, 4)$ and we have an exact sequence $0 \to E \to U_1 \oplus U_2 \to Q \to 0$, where E is a regular elementary H-module of dimension-vector $(1, 1, 0)$ and Q is also an elementary module with $\mathbf{dim}\, Q = (3, 5, 4) = \mathbf{dim}\, \tau^2 E$.

Theorem. *Let C be a connected wild hereditary algebra and M an indecomposable regular quasi–simple C-module with the property that $C[M]$ is tilted or concealed canonical. Then M is orbital elementary.*

Proof. Assume that $C[M]$ is a tilted algebra of type H. The proof can be reduced to the following case. There exists a tilting H-module $T = X \oplus P$ such that P is the projective generator of $X^\perp$ with $\operatorname{End}(P) = C$, X is quasi–simple regular, in the Auslander–Reiten sequence $0 \to \tau_H \to Z \to X \to 0$ we have that Z a is quasi–simple regular C-module and $A = C[Z]$ is tilted of type H.

In this situation we can define a functor $F : \operatorname{reg} C \to \operatorname{reg} H$ by the formula $F(M) := \tau_H^{-m} \tau_T^{2m} \tau_C^{-m} M$, where $m \gg 0$. The functor F is full and dense. If M is indecomposable then $F(M) = 0$ if and only if $M = \tau_C^i Z$ for some $i \in \mathbb{Z}$. We also have the following theorem.

Theorem. *Let $\eta : 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be a short exact sequence in* $\operatorname{reg} C$.

(a) *We have a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & F(U) & \xrightarrow{F(f)} & F(V) & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F(V) & \xrightarrow{F(g)} & F(W) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}.$$

(b) $F(\eta) = 0$ *if and only if for each* $\widetilde{Z} \in \operatorname{add}(\tau_C^i Z)$ *the morphism* $(\widetilde{Z}, g) : (\widetilde{Z}, V) \to (\widetilde{Z}, W)$ *is an epimorphism.*

Take now $\widetilde{Z} \in \operatorname{add}(\tau)C^i Z)$ and a short exact sequence $\eta : 0 \to U \to \widetilde{Z} \to W \to 0$ with U and W regular. Then $F(\eta) = 0$ and it follows from the above theorem that η splits. $\square$