Hereditary categories associated with quivers

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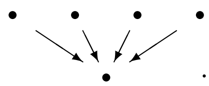
Let $\mathcal{C}$ be a noetherian abelian hereditary k-category, where k is algebraically closed field, with Hom and Ext^1-spaces finite dimensional over k. We assume also that $\mathcal{C}$ has nonzero projective objects and $\mathcal{C}$ has Serre duality $F : D^b(\mathcal{C}) \to D^b(\mathcal{C})$ such that $D\,\mathrm{Hom}(A, B) \simeq \mathrm{Ext}^1(B, FA)$.

Let Q be a finite quiver without oriented cycles. Then the category of finite dimensional representations of Q, which is equivalent to the category $\mathrm{mod}\,kQ$, satisfies above properties.

Consider the quiver



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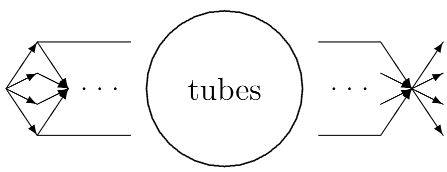
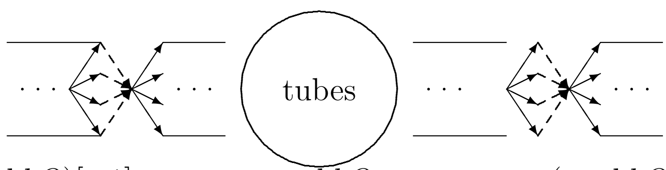
The Auslander–Reiten quiver of $\mathrm{mod}\,kQ$ has the form



The category $D^b(kQ)$ can be visualize as follows



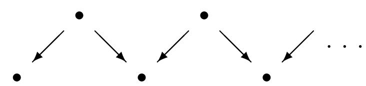
$(\mathrm{mod}\,kQ)[-1]$ $\mathrm{mod}\,kQ$ $(\mathrm{mod}\,kQ)[1]$

Moreover we have only new maps from $(\mathrm{mod}\,kQ)[i]$ to $(\mathrm{mod}\,kQ)[i+1]$ and they are given by Ext^1. We have the Auslander–Reiten translation τ which is an equivalence between nonprojective and noninjective modules. We have also a Nakayama functor N which an equivalence between projective and injective modules. The Serre duality is a combination of those two functors with Nakayama functor shifted by -1. Note that we have $D\operatorname{End}(P) \simeq \operatorname{Ext}^1(P, N(P)[-1]) \simeq \operatorname{Hom}(P, N(P))$ for a projective module P. We have also a natural map $g : PradN(P)$ for P indecomposable since $P/\operatorname{rad} P \simeq \operatorname{soc} N(P)$.

Theorem 1. *The category $\mathcal{C}$ has Serre duality if and only if $\mathcal{C}$ has almost split sequences, for each indecomposable projective module there is a maximal submodule* $\operatorname{rad} P$ *and* $P/\operatorname{rad} P$ *has injective envelope* I *in* $\mathcal{C}$ (*and all injective occurs*).

Let Q be an infinite quiver with no paths of infinite length. The the category of finite dimensional representations of Q has the same properties.

Consider the quiver Q of type $\mathbb{A}_\infty$ with the following orientation



The Auslander–Reiten quiver consists of a preprojective component of type $\mathbb{N}Q^{\mathrm{op}}$ and a preinjective components of type $(-\mathbb{N})Q^{\mathrm{op}}$.

Let $\mathcal{C}$ be a connected category satisfying above properties. We associate a quiver Q with $\mathcal{C}$. Vertices of the quiver Q correspond to the indecomposable projective objects in $\mathcal{C}$. For each indecomposable projective object P in $\mathcal{C}$ $\operatorname{rad} P$ is a direct sum of indecomposable projective objects $P_1, \ldots, P_n$. Note that each object in $\mathcal{C}$ has a finite decomposition into a direct sum of indecomposable objects since the category $\mathcal{C}$ is noetherian. Then we put arrows from a vertex associated with P to vertices associated with $P_1, \ldots, P_n$. We can define preprojective objects in a usual way.

Let $\mathcal{P}$ denote the category of projective objects and $\overline{\mathcal{P}} \subset \mathcal{C}$ the category of factors of $\mathcal{C}$. The category $\overline{\mathcal{P}}$ is equivalent to the category $\operatorname{rep} Q$ of finitely presented representations of Q.

Now we list the properties of $\mathcal{C}$.

If I is an indecomposable injective object then I has finite length. We know that $S := \operatorname{soc} I$ is simple. The module I/S is injective and hence is a direct sum of indecomposable injective modules, thus $\operatorname{soc}(I/S)$ has finite length. We can consider the sequence $0 \operatorname{soc} I \subset \operatorname{soc}^2 I \subset \cdots \subset I$ such that $\operatorname{soc}^i I/\operatorname{soc}^{i-1} I$ has a finite length. Since the category $\mathcal{C}$ is noetherian the claim follows.

The quiver Q is locally finite. Choose a vertex x in Q and denote by P_x the projective object in $\mathcal{C}$ corresponding to x. Obviously, by construction of Q, there is only a finite number of arrows starting at x. There is only a finite number of arrows ending at x. If follows from the fact that for each projective we have a corresponding injective module and the arrows ending at x corresponds to epimorphisms between indecomposable injective modules. Since the indecomposable injective modules are of finite length the claim follows.

The quiver Q has no infinite path ending at a vertex.

The category $\mathcal{C}$ is generated by the preprojective objects and $\mathcal{C}$ is uniquely determined by Q. Hence we will write $\mathcal{C} = \widetilde{\operatorname{rep}}\, Q$.

Theorem 2. *There exists $\mathcal{C}$ associated with Q if and only if Q is a "star", i.e. it is formed by quiver Q_0 where Q_0 has no infinite paths and is locally finite with "strings" (infinite paths starting at a vertex) attached to vertices of Q_0, only finite number for each vertex.*

Let Q be "star". We describe how to find the category $\mathcal{C}$. We know that $\operatorname{rep} Q \subset \mathcal{C}$. Consider $\mathbb{Z}Q^{\mathrm{op}}$ and choose a section Q_1^{op} such that $\mathbb{Z}Q_1^{\mathrm{op}} = \mathbb{Z}Q^{\mathrm{op}}$ and Q_1 has no infinite paths. Then the category $\mathcal{C}_1$ associated with Q_1 is just $\operatorname{rep} Q_1$, which is the same as the category of finite dimensional representations of Q_1.

Let Q be a quiver of type $\mathbb{A}_\infty$ with a linear orientation ("string"). We can choose Q_1 to be a "zig-zag" quiver considered before.

Denote by $\mathcal{T}$ the preinjective component in $\mathcal{C}_1$ and by $\mathcal{F}$ the remaining part of $\mathcal{C}_1$. The pair $(\mathcal{T}, \mathcal{F})$ is a split torsion pair in $\mathcal{C}_1$. In the derived category we get a hereditary category $\mathcal{T}[-1] \vee \mathcal{F}$ which has a component of the form $\mathbb{Z}Q^{\mathrm{op}}$. Now we construct a new split torsion pair $(\mathcal{T}', \mathcal{F}')$ according to the section in this component given by Q^{op} and we tilt again to form $\mathcal{T}'[-1] \vee \mathcal{F}'$. This is our desired category.

Note that in the example considered above the category $\operatorname{rep} Q$ consists only of direct sums of modules of finite length and of projective modules. Hence we only have a preinjective component and a line of projective in $\operatorname{rep} Q$.