Derived tameness

based on the talk by Christof Geiss

January 18, 2001

Let Λ be a finite dimensional algebra over an algebraically closed field k. We have the Happel's functor $F : D^b(\Lambda) \to \underline{\mathrm{mod}}\, \hat{\Lambda}$, where $\hat{\Lambda}$ is the repetitive algebra of Λ. The functor F is a full embedding and is an equivalence if and only if $\operatorname{gldim} \Lambda < \infty$.

Let X be an object of $D^b(\Lambda)$. We set $\mathbf{hdim}\, X$ to be $(\dim_K H^i(X))_{i \in \mathbb{Z}} \in \mathbb{N}^{(\mathbb{Z})}$.

Lemma. *If $G : D^b(\Lambda) \to D^b(\Gamma)$ is an equivalence then there is a linear estimate $\alpha : \mathbb{N}^{(\mathbb{Z})} \to \mathbb{N}^{(\mathbb{Z})}$ such that $\mathbf{hdim}\, GX \leq \alpha(\mathbf{hdim}\, X)$.* □

We say that Λ has a discrete derived category if for each $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$ there exists only finitely many isoclasses of $X \in D^b(\Lambda)$ with $\mathbf{hdim}\, X = \mathbf{h}$. Using the above lemma we get that this notion is stable under equivalence of derived categories.

Theorem (Vossieck)**.** *Let Λ be connected. The following conditions are equivalent.*

(a) *$D^b(\Lambda)$ is discrete.*

(b) *$D^b_{\mathrm{perf}}(\Lambda)$ is discrete.*

(c) *$\mathrm{mod}\, \hat{\Lambda}$ is discrete.*

(d) *Λ is piecewise hereditary of Dynkin type or Λ is gentle with one cycle and no clock condition.*

Piecewise hereditary algebras of Dynkin type are classified. They are simply connected algebras with positively defined Euler form. For type $\mathbb{A}_m$ we get gentle trees, type $\mathbb{D}_n$ has been described by Wenderlich/Keller, and $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ were listed by Roggon.

An algebra $A = KQ/I$ is called gentle if it satisfies the following conditions:

- I is generated by paths of length 2;
- at each vertex start and stop at most two arrows;
- for each arrow β there is at most one arrow α such that $\alpha\beta \notin I$ and at most one γ such that $\beta\gamma \notin I$;
- for each arrow β there is at most one arrow α such that $\alpha\beta \in I$ and at most one γ such that $\beta\gamma \in I$.

The clock condition is fulfilled if there the same number of clock and anticlockwise relations.

Proof. The implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are standard. The implication (b) $\Rightarrow$ (d) needs work. The implication (d) $\Rightarrow$ (c) is known. $\square$

The following definition of derived tameness has been suggested by Geiss and Krause. An algebra Λ is called derived tame if for each $\mathbf{h} \in \mathbb{N}^{(\mathbb{Z})}$ there exists a finite number of complexes $M_1, \ldots, M_n$ of Λ-$k[X]$-bimodules such that almost all indecomposable $X \in D^b(\Lambda)$ with $\mathbf{hdim}\, X = \mathbf{h}$ are isomorphic to some $M_i \otimes_{K[X]} K[X]/(X - \lambda)$. This definition satisfied the requirement of de la Peña that for an algebra Λ of finite global dimension derived tameness is equivalent to tameness of $\hat{\Lambda}$.

We have the following properties of derived tameness.

- If Λ is derived equivalent to Γ then Λ is derived tame if and only if Γ is derived tame. In the proof we need Rickard's theorem that there exists an equivalence of the form $T_\Lambda \otimes^{\mathbb{L}} - : D^b(\Lambda) \to D^b(\Gamma)$. It is not needed for "generic" definition of tameness, but in the case $\operatorname{gldim} \Lambda = \infty$ it is not clear if this definition is equivalent to the above one.
- If $\hat{\Lambda}$ is tame then Λ is derived tame. We use that the Happel's functor is "open". It means particularly that if $\tau_{\hat{\Lambda}} M \simeq M$ then M is in image of the Happel's functor. Obviously, if $\operatorname{gldim} \Lambda < \infty$ then Λ derived tame implies $\hat{\Lambda}$ is tame.

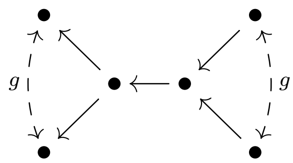
Examples of derived tame algebras are piecewise hereditary algebras of Euclidean and tubular type, gentle algebras, skewed gentle algebras, tree algebras with nonnegative Euler form (Brüstle/Geiss).

Proposition. *Let Λ be a gentle algebra which is not derived equivalent to $\mathbb{A}_n$ or $\tilde{\mathbb{A}}_n$. Then $\Gamma(\underline{\operatorname{mod}}\, \hat{\Lambda})$ contains a finite number of components of type $\mathbb{Z}\mathbb{A}_\infty$.*

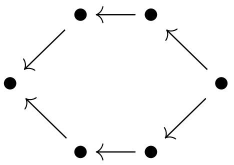
Gentle algebras are closed under derived equivalence. In the proof we use that special biserial algebras are preserved under stable equivalence.

Let Λ be an algebra. Assume that $\operatorname{char} k \neq 2$ and there exists an automorphism g of Λ of order 2 such that the skew group algebra $\Lambda * \langle g \rangle$ is Morita equivalent to a special biserial algebra.

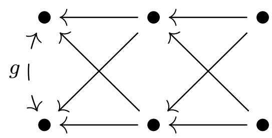
For example we get from the path algebra of the quiver



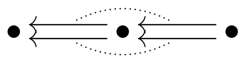
we get the path algebra of the quiver



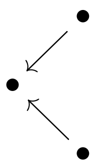
Similarly, for the incidence algebra of the poset



we get an algebra of the following bounded quiver

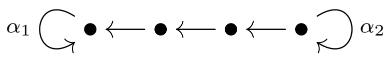


We present some more general setup for this construction. Recall that we have the following isomorphism of algebras $k[X]/(X^2 - X) \simeq K \times K$. Similarly, $A := k[0 \xleftarrow{\alpha} 1 \circlearrowleft \varepsilon\,]/(\varepsilon^2 - \varepsilon)$ is isomorphic to the path algebra of the quiver

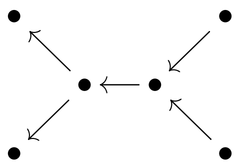


Note that given an A-module M we have that $M(\varepsilon)$ splits and $M(1) \simeq \operatorname{Ker} M(\varepsilon) \oplus \operatorname{Im} M(\varepsilon)$.

Let KQ/I be a string algebra. Choose loops $\varepsilon_1, \ldots, \varepsilon_n$ in Q with $\varepsilon_i^2 \in I$ and ε_i not start of other relations. We replace ε_i^2 by $\varepsilon_i^2 - \varepsilon_i$ and obtain an algebra Λ. We have also an automorphism g of order 2 which interchanges $1_{\varepsilon_i} - \varepsilon_i$ with ε_i and $\Lambda * \langle g \rangle$ is a string algebra if $\operatorname{char} k \neq 2$. For example starting from the quiver



bounded by α_1^2 and α_2^2, we obtain by replacing α_i^2 by $\alpha_i^2 - \alpha_i$ an algebra isomorphic to the path algebra of the quiver



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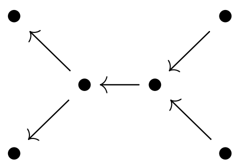
By work of Crawley–Boevey and Deng we know that the representation theory of Λ is independent of $\operatorname{char} k$. By Geiss also description of homomorphism is independent of $\operatorname{char} k$. Hence we may "assume" $\operatorname{char} k \neq 2$ and use theory of Reiten–Riedtmann to calculate Auslander–Reiten quiver.

If we start with KQ/I gentle then $\Lambda * \langle g \rangle$ is gentle. Thus $\hat{\Lambda} * \langle g \rangle$ is special biserial and $\hat{\Lambda}$ modulo socle is clannish. We called such algebras Λ skewed gentle algebras. Skew gentle algebras are derived tame. However, they are not closed under derived equivalence, since the path algebr of the quiver



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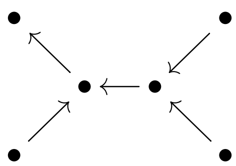
which is skew gentle, is derived equivalent to the path algebra of the quiver



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which is not skew gentle.

It is conjectured by the author that if Γ is a (connected) derived tame algebra then either Γ is either piecewise hereditary of tame type or derived equivalent to a skewed gentle algebra. Some evidence is the following fact.

Theorem (Brüstle, Geiss)**.** *A tree algebra Γ is derived tame if and only if χ_Γ is positive semidefined. If Γ is derived tame then it is derived equivalent* *either to some hereditary or canonical algebra of tame type or it is derived equivalent to the incidence algebra of the following poset*