Derived subspace problem

based on the talk by Christof Geiss
(joint work with Bernhard Keller)

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Dg-algebra is a graded algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ together with a differential $d : A \to A$, i.e. a linear map $d : A \to A$ of degree 1 such that $d^2 = 0$ and $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for any homogeneous elements a and b of A. If A is a dg-algebra then by (right) dg-module over A we mean a graded A-module M with a differential d such that $d(ma) = d(m)a + (-1)^{|m|} m d(a)$. By $\mathcal{C}(A)$ we will denote the category of dg-modules over A. We will consider ordinary algebras as dg-algebras concentrated in degree 0. In that case $\mathcal{C}(A)$ is the category of complexes of (ordinary) modules over A.

For a dg-algebra A and $M \in \mathcal{C}(A)$ we consider a dg-algebra

$$A[M] := \begin{bmatrix} k & M \\ 0 & A \end{bmatrix}$$

with differential coming from each components. Objects in $\mathcal{C}(A[M])$ are given by triples (V, X, f) with $V \in \mathcal{C}(k)$, $X \in \mathcal{C}(A)$ and $f \in \mathrm{Hom}_{\mathcal{C}(A)}(V \otimes_k M, X)$. Sometimes we represent f by it's adjoint $\tilde{f} \in \mathrm{Hom}_{\mathcal{C}(k)}(V, \mathcal{H}om_A(M, X))$.

We have also the derived subspace problem $\mathcal{U}(D(A), M)$ whose objects are (V, X, f) with $V \in \mathcal{D}(k)$, $X \in D(A)$ and $f \in \mathrm{Hom}_{\mathcal{D}(k)}(V, \underline{R}\,\mathrm{Hom}_A(M, X))$, where $\underline{R}\,\mathrm{Hom}_A(M, X) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_A(M, X))$. We have the following connection between the above notions.

Proposition (reduction)**.** *The functor $\mathcal{P}^M_u : \mathcal{D}(A[M]) \to \mathcal{U}(\mathcal{D}(A), M)$, $(V, X, f) \mapsto (V, X, \tilde{f})$ is full, dense and reflects isomorphisms.*

Note that $\mathcal{P}^M_u$ is not faithful in general.

Proposition (compatibility)**.** *"Reduction is compatible with standard equivalences",i .e. of type $- \otimes^{\mathbb{L}}_A T_B$.*