Nonpolynomial growth tree algebras

based on the talk by Thomas Brüstle (Bielefeld)

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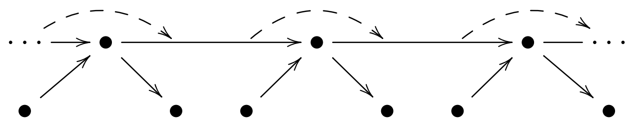
Strongly simply connected tame algebras can be divided into two disjoint classes. First one is formed by polynomial growth strongly simply connected algebras for which supports of indecomposable modules are known. They can be either tame tilted or coil algebras. Supports of one-parameter families are tame concealed (critical) or tubular algebras. The other class consists of nonpolynomial growth algebras. Here supports of one-parameter families can have arbitrary global dimension as the following example shows



The main aim of this talk is to show that a strongly simply connected algebra A is tame if and only if the Tits form q_A of A is weakly nonnegative. It is known that q_A is weakly nonnegative if and only if there exists no convex subcategory C of A which is hypercritical. Moreover, A is tame of polynomial growth if and only if there exists no convex subcategory C of A which is either hypercritical or pg-critical. Thus we have to show that if q_A is weakly nonnegative and there exists a convex subcategory C of A which is pg-critical then A is tame.

We have the following two important examples of nonpolynomial growth tame algebras. First class of examples is formed by string algebras. Locally they are extensions or coextension by simple regular modules of a hereditary algebras of type $\tilde{\mathbb{A}}_n$, which lead to tubular type (p, q). Each module can be used only once for extension and only once for coextension. Moreover we do not allow extensions and coextensions by modules of regular length two.

Another class if formed by nonpolynomial growth sincere tame tree algebras. Here the critical subcategories are hereditary algebras of type $\tilde{\mathbb{D}}_n$. We allow extensions and coextensions by simple regular modules from a fixed tube of rank $n-2$. We can extend and coextend by a simple regular module twice and we can use modules of regular length two from the give tube.

Theorem. *Let A be a tree algebra. Suppose A is sincere with weakly nonnegative Tits form and contains pg-critical convex subcategory. Then A degenerates to a gentle algebra. In particular, A is tame.*

As the consequence of the above theorem we obtain that a tree algebra is tame if and only if its Tits form is weakly nonnegative. Moreover, we know that sincere tame tree algebras belong to one of the following three classes: tame tilted algebras, coil algebras and deformations of gentle algebras.

The proof of the theorem consists of three step. First, we define a class $\mathbb{D}$-alg of algebras. Second, we show that if A satisfies conditions of the theorem then A is a quotient of an algebra from $\mathbb{D}$-alg. Finally, we prove that all algebras in $\mathbb{D}$-alg degenerate to gentle algebras.

In order to define $\mathbb{D}$-alg we recall pg-critical tree algebras. They can be written in one of the following forms: $\tilde{D}[N, D]$, $\tilde{D}[N][N]$, $\tilde{D}[M]$ and dual ones, where $\tilde{D}$ is a hereditary algebra of type $\tilde{\mathbb{D}}$, N is a simple regular $\tilde{D}$-module, M is a $\tilde{D}$-module of regular length two and D is a hereditary algebra of type $\mathbb{D}$.

We define the class $\mathbb{D}$-alg by the following inductive definition:

(1) All pg-critical algebras belong to $\mathbb{D}$-alg.

(2) Suppose A belongs to $\mathbb{D}$-alg. Then A contains a pg-critical convex subcategory Λ and Λ is an appropriate extension or coextension of a hereditary convex subcategory $\tilde{D}$ of type $\tilde{\mathbb{D}}$. Let Λ' be of the one of the following forms: $\tilde{D}[N_1, D_1]$, $\tilde{D}[N_1][N_1]$, $\tilde{D}[M_1]$, or $\tilde{D}[N_1, B]$, where N_1 is a simple regular $\tilde{D}$-module, M_1 a $\tilde{D}$-module of regular length two, D_1 is a hereditary category of type $\mathbb{D}$ and B is a branch. If the following conditions are satisfied:

- The module N_1 (respectively M_1) belongs to the same tube $\mathscr{T}$ of $\Gamma(\operatorname{mod} \tilde{D})$ as the module connected with extension or coextension which leads to Λ.
- If L in $\mathscr{T}$ satisfies $\tilde{D}[L] \subset A$ then the modules L and N_1 (respectively M_1) are orthogonal.

then $A' = A \coprod_{\tilde{D}} \Lambda'$ belongs to $\mathbb{D}$-alg. We have also the dual operation for coextensions.

To show that each algebra from $\mathbb{D}$-alg degenerates to a gentle one, we use the following basic observations. First note the the path algebra A of the quiver

$$\begin{tikzcd}
\bullet \arrow[dr] & \\
& \bullet \\
\bullet \arrow[ur] &
\end{tikzcd}$$

is isomorphic to the path algebra of the quiver Q of the form

$$\begin{tikzcd}
\bullet \arrow[loop left, "e"] \arrow[r] & \bullet
\end{tikzcd}$$

bound by $e = e^2$. Let $A_\lambda = KQ/(e^2 - \lambda e)$. Then $A_\lambda \simeq A$ for $\lambda \neq 0$ and A_0 is gentle.

Similarly, the path algebra A of the quiver

$$\begin{tikzcd}
\bullet \arrow[dr] & \\
& \bullet \arrow[dl] \\
\bullet &
\end{tikzcd}$$

is isomorphic to the path algebra of the quiver Q of the form

$$\begin{tikzcd}
\bullet \arrow[dr, "a"] \arrow[dd, "x"'] & \\
& \bullet \arrow[dl, "b"] \\
\bullet &
\end{tikzcd}$$

bound by $x = ab$. Let $A_\lambda = KQ/(ab - \lambda x)$. Then $A_\lambda \simeq A$ for $\lambda \neq 0$ and A_0 is gentle.

Finally, consider the path algebra A of the quiver

$$\begin{tikzcd}
\bullet \arrow[d, "a"] & & \\
\bullet \arrow[r, "b"] & \bullet \arrow[r, "c"] & \bullet
\end{tikzcd}$$

bound by $abc = 0$. Then A is isomorphic to the path algebra of the quiver Q of the form

$$\begin{tikzcd}
\bullet \arrow[d, "a"] \arrow[dr, "x"] & & \\
\bullet \arrow[r, "b"] & \bullet \arrow[r, "c"] & \bullet
\end{tikzcd}$$

bound by $x = ab$ and $xc = 0$. Let $A_\lambda = KQ/(ab - \lambda x)$. Then $A_\lambda \simeq A$ and A_0 is gentle.