Tame tree algebras and integral quadratic forms

based on the talk by Thomas Brüstle (Bielefeld)

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Let k be a fixed algebraically closed field. By an algebra we mean a finite dimensional algebra over k. Usually we assume that considered algebras are path algebras of bound quivers. Given an algebra A we may define two quadratic forms on $K_0(A)$: the Euler form χ_A and the Tits form q_A. The Euler from is well defined if $\operatorname{gl.dim} A < \infty$ and it is given by

$$\chi_A([X]) := \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_A^i(X, X).$$

For the Tits form we have

$$q_A([X]) := \sum_{i=0}^{2} (-1)^i \dim_K \operatorname{Ext}_A^i(X, X).$$

Theorem (Gabriel). *Assume A is the path algebra of the quiver Q.*

(1) *A is representation-finite.*

(2) *χ_A is positive.*

(3) *Q is a disjoint union of Dynkin quivers.*

Theorem (Ringel–Dlab). *Let A be a path algebra of a connected quiver Q. The following conditions are equivalent.*

(1) *A is tame, but not representation finite.*

(2) *χ_A is nonnegative, but not positive.*

(3) *Q is an Euclidean quiver.*

By $\operatorname{mod} A$ we will denote the category of finite dimensional modules over A and $D^b(A)$ denotes the bounded derived category of $\operatorname{mod} A$. If A and B are algebras such that $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories then the Euler forms χ_A and χ_B are $\mathbb{Z}$-equivalent, i.e. there exists a $\mathbb{Z}$-invertible matrix T such that $\chi_A = \chi_B T$.

The quadratic forms $p : \mathbb{Z}^n \to \mathbb{Z}$ of the form

$$p(x) = \sum_{i=1}^{n} x_i^2 + \sum_{1 \le i < j \le n} p_{i,j} x_i x_j$$

will be called unit forms. The $\mathbb{Z}$-equivalence classes of positive unit forms correspond to Dynkin diagrams.

Let p be a nonnegative unit form. Then $\operatorname{rad} p$, which is by definition the set of all x such that $p(x) = 0$, is a subgroup of $\mathbb{Z}^n$. We have an induced positive unit form $\overline{p} : \mathbb{Z}^n / \operatorname{rad} p \to \mathbb{Z}$, where $\mathbb{Z}^n / \operatorname{rad} p \simeq \mathbb{Z}^{n-c}$ and c is called the corank of p. We know that $\overline{p}$ is positive, hence is $\mathbb{Z}$-equivalent to the form of disjoint union of Dynkin diagrams.

Theorem (Barrot–de la Peña)**.** *The unit form p is uniquely determined, up to $\mathbb{Z}$-equivalence, by its corank and the Dynkin type of $\overline{p}$.*

The forms with corank 1 are given by Euclidean diagrams and there is no quiver Q with nonnegative form and corank at least 2.

An algebra $A = kQ/I$ is called a tree algebra if Q is a tree. Tree algebras are uniquely determined by q_A.

Theorem. *Let A be a tree algebra. Then A is tame if and only if $q_A(\mathbb{N}^n) \ge 0$. Moreover, $D^b(A)$ is tame if and only if χ_A is nonnegative.*

For any $c \in \mathbb{N}$ there are tree algebras A with nonnegative Euler form such that corank of χ_A equals c.

Let Δ be a Dynkin quiver. We have the corresponding representation finite algebra $k\Delta$. Moreover, the positive unit forms are classified by Dynkin diagrams. On the other hand, we have a semisimple complex Lie algebras $\mathfrak{g}_\Delta$ corresponding to Δ. It is known that $\mathfrak{g}_\Delta = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ with $\mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, where $\Phi^+ = \chi_A^{-1}(1)$ is the set of dimension vectors of indecomposable $k\Delta$-modules. We can introduce the Lie bracket $[-,-]$ in $\mathbb{C}\Phi^+$ such that it gives a Lie algebra isomorphic to $\mathfrak{g}^+$.

Let $A = KQ/I$ be a finite dimensional algebra. Let $S_1, \ldots, S_n$ be the complete set of representatives of simple A-modules and Ω be the set of words in $\{S_1, \ldots, S_n\}$. We define a ring structure on $\mathbb{C}\Omega$ by multiplication coming from the concatenation of words. Let R be an ideal consisting of all $f \in \mathbb{C}\Omega$

such that $f(M) = 0$ for any $M \in \operatorname{mod} A$. Here, for $f = \sum_{\omega\in\Omega} c_\omega \omega$ we have $f(M) := \sum_{\omega\in\Omega} c_\omega \omega(M)$, where $\omega(M)$, with $\omega = S_{i_0} \cdots S_{i_{k-1}}$, is the Euler–Poincare characteristic of the variety of filtrations $0 = M_k \subset \cdots \subset M_0 = M$ such that $M_j/M_{j+1} \simeq S_{i_j}$. Let $\mathscr{C}_A$ be the Lie subalgebra of $(\mathbb{C}\Omega/R, [-,-])$ with $[f, g] := fg - gf$, generated by $S_1, \ldots, S_n$.

Theorem (Ringel, 1990). *Let A be the path algebra of a Dynkin quiver Δ. Then $\mathfrak{g}^+_\Delta$ is isomorphic to $\mathscr{C}_A$ as a Lie algebra.*

It has been proved by Frenkel–Malkin–Vybornov using reflection functors.

Theorem (Frenkel–Malkin–Vybornov). *If A is the path algebra of an Euclidean quiver Δ, then $\mathfrak{g}^+_\Delta$ is isomorphic to $\mathscr{C}_A$ as a Lie algebra, where $\mathfrak{g}^+_\Delta$ is a positive part of the affine Kac–Moody algebra, which is given by generators $S_1, \ldots, S_n$ and Serre relations, that is $[S_i, S_j] = 0$ if there is no arrow between i and j and $[S_i, [S_i, S_j]] = 0$ if there is exactly one arrow between i and j.*

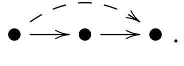
Recently Saito described the elliptic Lie algebras in terms of generators and relations. Let q be a nonnegative unit form and $\Phi := q^{-1}(1)$ be a generalized root system. Saito has classified the root systems when corank of q is 2. There is a Lie algebra $\mathfrak{g}(\Phi)$ constructed by Bochards (as vertex algebra). In the "elliptic" case of corank 2 Saito has describe it also in terms of generators and relations. The generators are simple modules and relations are Serre relations together with some additional relations.

Consider the path algebra A of the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet\,.$$

Then we have $[S_1, S_3](M) = S_1S_3(M) - S_3S_1(M)$. Note $S_1S_3(M) \neq 0$ implies M has composition factors S_1 and S_3, hence $M = S_1 \oplus S_3$ and $[S_1, S_3] = 0$. Similarly, $[S_2, S_3](M) = S_2S_3(M) - S_3S_2(M)$. Thus we need to consider $M = S_2 \oplus S_3$ and $M = P_2$, and we get $[S_2, S_3](M) = \delta_{M,P_2}$. Finally, $[S_2, [S_2, S_3]](M) = 0$ and we only have to consider $M = S_2 \oplus P_2$.

Let B be the path algebra of the following bound quiver



Then besides Serre relations we also have $[S_1, [S_2, S_3]] = 0$.

It is conjectured by the author that if A is a tubular alebra, then the positive part $\mathfrak{g}^+(A)$ of Borchard's Lie algebra is isomorphic to $\mathscr{C}_A$.