The automorphism groups of domestic or tubular exceptional curves over the real numbers

based the talk by Dirk Kussin (Paderborn)

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Let k be a field. We will consider tame hereditary and canonical k-algebras (in the sense of Ringel and Crawley-Boevey). The structure of the category of indecomposable modules over such an algebra Λ is well-known. It particular, it known there exists a separating tubular family $\operatorname{ind}_0 \Lambda$ which is indexed by some set $\mathbb{X}$. Our aim is to study the geometry of $\mathbb{X}$.

The easiest case is so called homogeneous case, when

$$\Lambda = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix},$$

where F and G are skew fields, which are finite dimensional over k, and M is a tame F-G-bimodule with k acting centrally. In this case all tubes are homogeneous.

If k is algebraically closed then $M = k \oplus k$ and $\mathbb{X} = \mathbb{P}^1(k)$. In nonhomogeneous cases for k algebraically closed we obtain weighted projective lines introduced by Geigle and Lenzing.

For $k = \mathbb{R}$ the structure of $\mathbb{X}$ as a topological space has been described by Dlab and Ringel. However, the geometry of $\mathbb{X}$ in general is not understood. It has been implicitly described in terms of exceptional curves by Lenzing. We present here the explicit description for $k = \mathbb{R}$. This description will be done in terms of the automorphism group.

Let $\mathscr{H} = \operatorname{coh} \mathbb{X}$ be the category of coherent sheaves over $\mathbb{X}$. It is a hereditary noetherian category with Serre duality. We define $\operatorname{Aut} \mathscr{H}$ to be the class of all auto-equivalences $F : \mathscr{H} \to \mathscr{H}$ modulo isomorphism relation. We put $\operatorname{Aut} \mathbb{X}$ to be the subgroup of $\operatorname{Aut} \mathscr{H}$ consisting of all $F \in \operatorname{Aut} \mathscr{H}$ which fix the structure sheaf L.

It is known that each exceptional curve $\mathbb{Y}$ arise by "insertion of weights" from a homogeneous curve $\mathbb{X}$. We write $\mathbb{Y} = \mathbb{X}\left(\begin{smallmatrix} p_1 & \cdots & p_t \\ x_1 & \cdots & x_t \end{smallmatrix}\right)$, where $x_1, \ldots, x_t \in \mathbb{X}$ are pairwise distinct and $p_1, \ldots, p_t > 1$ are weights.

Lemma. *If $\mathbb{X}$ and $\mathbb{Y}$ are as above, then $\operatorname{Aut}\mathbb{Y}$ is the subgroup of $\operatorname{Aut}\mathbb{X}$ formed by the automorphisms which preserve the weights (i.e. $p(Fy) = p(y)$ for all $y \in Y$).*

Consider the homogeneous case $\Lambda = \left(\begin{smallmatrix} G & 0 \\ M & F \end{smallmatrix}\right)$. Assume that $\operatorname{ind}\Lambda$ consists of the preprojective component, the family of homogeneous tubes $\operatorname{ind}_0 \Lambda$ and the preinjective component. Then the category $\operatorname{coh}\mathbb{X}$ consists of the transinjective component build up from vector bundles and the tubes consisting from objects of finite length. Let $0 \to L \to \overline{L} \to \tau^{-}L \to 0$ be the Auslander–Reiten sequence, where L is the structure sheaf of $\mathbb{X}$. Then $\overline{L}$ is indecomposable. Moreover, $\operatorname{Hom}(L, \overline{L}) = M$, $\operatorname{End}(L) = G$ and $\operatorname{End}(\overline{L}) = F$.

We define the group $\operatorname{Aut} M = \operatorname{Aut}_k({}_F M_G)$ to be the set of all triples $\varphi = (\varphi_F, \varphi_M, \varphi_G)$, where φ_F is a k-automorphism of F, φ_G is a k-automorphism of G and $\varphi_M : M \to M$ is a k-linear bijection such that $\varphi_M(fmg) = \varphi_F(f)\varphi_M(m)\varphi_G(g)$. Equivalently, we may define $\operatorname{Aut}(M)$ as the the group of k-autoequivalences of the category $\{L, \overline{L}\}$.

A triple $\varphi = (\varphi_F, \varphi_M, \varphi_G) \in \operatorname{Aut} M$ is called inner if there exists a unit f in F and an unit g in G such that $\varphi_F(x) = f^{-1}xf$, $\varphi_G(y) = g^{-1}xg$ and $\varphi_M(m) = f^{-1}mg$. We denote the group of inner automorphisms by $\operatorname{Inn} M$. Each triple $(\varphi_F, \varphi_M, \varphi_G)$ induces the automorphism of the k-algebra Λ in a natural way. This automorphism is inner in the usual way if the triple is inner. As usual we put $\operatorname{Out} M := \operatorname{Aut} M / \operatorname{Inn} M$.

Lemma. *Let $\mathbb{X}$ be a homogeneous exceptional curve with underlying tame bimodule M. Then $\operatorname{Aut}\mathbb{X}$ is isomorphic to $\operatorname{Out} M$.*

Proof. Given an automorphism F of $\mathbb{X}$ we have it is given by an equivalence $F : \mathscr{H} \to \mathscr{H}$ fixing L. Then $\overline{L}$ is also fixed. Hence $F|_{\{L,\overline{L}\}}$ is an autoequivalence of $\{L, \overline{L}\}$, hence belongs to $\operatorname{Aut} M$. Moreover, $F \simeq 1_{\mathscr{H}}$ if and only if its restriction is an inner automorphism.

Conversely, given an autoequivalence $F : \{L, \overline{L}\} \to \{L, \overline{L}\}$ we have an induced element $\tilde{F} \in \operatorname{Aut}(\Lambda)$. Hence we get an equivalence $\tilde{\tilde{F}} : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ which extends to the derived category and by restriction we obtain a selfequivalence of $\operatorname{coh}\mathbb{X}$. Moreover this equivalence fixes L and F is inner if and only if $\tilde{\tilde{F}}$ is isomorphic to $1_{\operatorname{mod}\Lambda}$. The above defined maps are mutually inverse. □

From now we assume $k = \mathbb{R}$. Let $\mathbb{X}$ be a homogeneous exceptional curve over $\mathbb{R}$. We have up to duality five cases.

	M	$\operatorname{Out} M$	R
1	${}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$	$\mathrm{SO}_3(\mathbb{R})$	$\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2)$
2	${}_{\mathbb{R}}(\mathbb{R}\oplus\mathbb{R})_{\mathbb{R}}$	$\mathrm{PGL}_2(\mathbb{R})$	$\mathbb{R}[X,Y]$
3	${}_{\mathbb{C}}(\mathbb{C}\oplus\mathbb{C})_{\mathbb{C}}$	$\mathrm{PGL}_2(\mathbb{C})\rtimes\mathbb{Z}_2$	$\mathbb{C}[X,Y]$
4	${}_{\mathbb{H}}(\mathbb{H}\oplus\mathbb{H})_{\mathbb{H}}$	$\mathrm{PGL}_2(\mathbb{R})$	$\mathbb{H}[X,Y]$, X, Y are central
5	${}_{\mathbb{C}}(\mathbb{C}\oplus\overline{\mathbb{C}})_{\mathbb{C}}$	$\mathbb{R}_+\rtimes\mathbb{Z}_2\rtimes\mathbb{Z}_2$	$\mathbb{C}[X,\overline{Y}]$

In each case $\operatorname{coh}\mathbb{X}\simeq\operatorname{mod}^{\mathbb{Z}}(R)/\operatorname{mod}_0^{\mathbb{Z}}(R)$.

Let $\mathbb{X}$ be the projective spectrum $\operatorname{Proj}(R)$ of R. All homogeneous primes ideals of height 1 in R are of the form $R\pi=\pi R$ with π homogeneous. We list the possible forms of π in all cases.

1. We have $\pi=ax+by+cz$, where $(a,b,c)\neq(0,0,0)$. Hence $\mathbb{X}$ can be identified with $S^2/\pm 1\simeq\mathbb{P}^1(\mathbb{C})/\mathbb{Z}_2$, where $\mathbb{Z}_2=\langle z\mapsto -1/\overline{z}\rangle$. Here, all points are complex, that is for each $x\in\mathbb{X}$ we have $\operatorname{End}(S_x)=\mathbb{C}$, where S_x is the simple sheaf concentrated in x.

2. We have the following possible forms of π:

   - X, $Y+\alpha X$, $\alpha\in\mathbb{R}$, real points;
   - $(Y+zX)(Y+\overline{z}X)$, $z\in\mathbb{C}\setminus\mathbb{R}$, complex points.

   Hence $\mathbb{X}\simeq\mathbb{P}^1(\mathbb{C})/\langle\bar{\cdot}\rangle$.

3. We have $\pi=X$ or $\pi=Y+zX$, $z\in\mathbb{C}$, and $\mathbb{X}=\mathbb{P}^1(\mathbb{C})$ is the Riemann sphere with complex points.

4. We have the following possible forms of π:

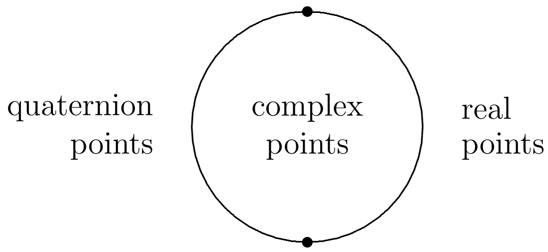
   - X, $Y+\alpha\mathbb{R}$, $\alpha\in\mathbb{R}$, quaternion points;
   - $(Y+zX)(Y+\overline{z}X)$, $z\in\mathbb{C}\setminus\mathbb{R}$, complex points.

   Hence $\mathbb{X}\simeq\mathbb{P}^1(\mathbb{C})/\langle\bar{\cdot}\rangle$.

5. We have the following possible forms of π:

   - X, Y, complex points.
   - $Y^2-\alpha X^2=(Y-\sqrt{\alpha}X)(Y+\sqrt{\alpha}X)$, $\alpha>0$, real points;
   - $Y^2-\alpha X^2$, $\alpha<0$, quaternion points;
   - $(Y^2-zX^2)(Y^2-\overline{z}X^2)$, $z\in\mathbb{C}\setminus\mathbb{R}$, complex points.

Hence $\mathbb{X}$ is a disk with the following distribution of points



Let Σ be the Riemann sphere. Then $\mathbb{X} = \Sigma$ or $\mathbb{X} = \Sigma/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by an antiholomorphic involution having fixed points of different type (in cases 2, 4, 5), or no fixed points (case 1).

Let $\operatorname{Aut}' \mathbb{X}$ be the group of conformal maps of Σ, which in cases different from 3 commute with the involution and preserve type of points. Recall that all conformal maps on Σ are given by Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \text{ or } z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d},$$

where $ab - bc \neq 0$. Hence the group of conformal maps is $\mathrm{PGL}_2(\mathbb{C}) \rtimes \mathbb{Z}_2$. Thus $\operatorname{Aut}' \mathbb{X}$ is:

1. $\mathrm{SO}_3(\mathbb{R})$.
2. $\mathrm{PGL}_2(\mathbb{R})$.
3. $\mathrm{PGL}_2(\mathbb{C}) \rtimes \mathbb{Z}_2$.
4. $\mathrm{PGL}_2(\mathbb{R})$.
5. $\mathbb{R}_+ \rtimes \mathbb{Z}_2$, where $\mathbb{R}_+ = \{z \mapsto \alpha z \mid \alpha > 0\}$.

Note that each $\varphi \in \operatorname{Aut} \mathbb{X}$ "permutes" points of $\mathbb{X}$.

Theorem. *By "restriction to points" we get the homomorphism of groups*

$$\Phi : \operatorname{Aut} \mathbb{X} \to \operatorname{Aut}' \mathbb{X},$$

which in cases 1–4 is an isomorphism, and in case 5 is a split epimorphism with kernel generated by γ.

In case 1 $\operatorname{Aut} \mathbb{X} = \mathrm{SO}_3(\mathbb{R})$. Mean geometry of $\mathbb{X}$ is equipped with additional metric structure (angles). As topological space $\mathbb{X}$ is just $\mathbb{P}^2(\mathbb{R})$, but its automorphism group is $\mathrm{PGL}_3(\mathbb{R})$.

Theorem. *If $\mathbb{X}$ is a tubular exceptional curve then there is an exact sequence of groups*

$$1 \to \operatorname{Pic}_0 \mathbb{X} \rtimes \operatorname{Aut} \mathbb{X} \longrightarrow \operatorname{Aut} D^b\mathbb{X} \to V \longrightarrow 1,$$

where V is either the breid group of B_3, or it is a subgroup of B_3 of index 3. If $B_3 = \langle s, l \mid sls = lsl \rangle$, then $V = \langle l^n, s \rangle$, where $n = 1$ or $n = 2$. If $n = 2$ then $\langle l^2, s \rangle = \langle l^2, s \mid (l^2 s)^2 = (sl^2)^2 \rangle$.

We obtain the $\mathbb{X}$ domestic means no parameter and if $\mathbb{X}$ is tubular then $\operatorname{Aut} \mathbb{X}$ is finite.