# Piecewise hereditary one-point extensions of wild hereditary algebras

based on the talk by Otto Kerner (Düsseldorf)

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Let $H$ be a connected wild hereditary algebra and $M$ a regular $H$-module such that $H[M]$ is piecewise hereditary. If $H[M]$ is quasi-tilted then $H[M]$ is piecewise hereditary. Moreover, in this case $H[\tau_H^{-r} M]$ is piecewise hereditary for $r > 0$. In addition, $H[\tau_H^{-r} M]$ is not quasi-tilted for $r \gg 0$. Finally, if $N$ is a regular $H$-module such that $H[N]$ is piecewise hereditary then there exists $r > 0$ such that $H[\tau^r N]$ is quasi-tilted.

Let $\mathscr{H}$ be a hereditary category and $X$ be a simple regular object from a $\mathbb{Z}\mathbb{A}_\infty$-component. Then $X^\perp \simeq \operatorname{mod} C$. If $Z \to X$ is an irreducible map then $C[Z]$ is a quasi-tilted algebra of type $\mathscr{H}$. Let $A = C[\tau_C^{-r} Z]$.

**Theorem.** *Let $A = C[\tau_C^{-r} Z]$ be a piecewise hereditary algebra which is not quasi-tilted. Then:*

(a) *$\mathscr{P}(C)$ is the unique preprojective component $\mathscr{P}(A)$ of $\Gamma(A)$;*

(b) *there exists connected wild concealed factor algebra $D$ of $A$ such that the unique preinjective component $\mathscr{I}(A)$ of $\Gamma(D)$ equals $\mathscr{I}(D)$;*

(c) *if $\mathscr{C}$ is a connected component of $\Gamma(A)$ which is neither preprojective nor preinjective then:*

  (i) *the stable part $\mathscr{C}^s$ of $\mathscr{C}$ is of the form $\mathbb{Z}\mathbb{A}_\infty$;*

  (ii) *if $M \in \mathscr{C}^s$ then $\tau_A^{-m} M$ is a $C$-module and $\tau_C^{-r} \tau_A^{-m} M = \tau_A^{-m-r} M$ for $m \gg 0$ and $r > 0$;*

  (iii) *if $M \in \mathscr{C}^s$ then $\tau_A^m M$ is a $D$-module for $m \gg 0$;*

(d) *If $N$ is an indecomposable regular $C$-module, then $\tau_A^{-r} \tau_C^{-m} N = \tau_C^{-m-r} N$ for $m \gg 0$ and $r > 0$. Dually, if $N'$ is an indecomposable regular $D$-module, then $\tau_A^r \tau_D^m N' = \tau_D^{m+r} N'$ for $m \gg 0$, $r > 0$.*

Let $T_0 := \tau_C^{r-1} DC$, where $DC$ is a minimal injective cogenerator of $X^{\perp}$. There exists $T_0'$ in $\operatorname{add} T_0$ such that we have a minimal approximation $\lambda : \tau_{\mathscr{H}} X \to T_0'$. Then $\lambda$ is injective and $T_1 := \operatorname{Coker} \lambda$ is an indecomposable object such that $T := T_0 \oplus T_1$ is a tilting object. We define $B := \operatorname{End}(T)$.

One can show that $\Gamma(B)$ has a unique preinjective component $\mathscr{I}(D')$, where $D'$ is a a wild concealed algebra. Moreover, if $\mathscr{C}$ is a component contained in $\mathscr{H}(T) := \operatorname{Ext}(T, \mathscr{F}(T))$ different from $\mathscr{I}(D')$ then $\mathscr{C}^s = \mathbb{Z}\mathbb{A}_\infty$ and for $M \in \mathscr{C}^s$ we have that $\tau_B^m M$ is a $D'$-module for $m \gg 0$. Moreover, if $N$ is a regular $D'$-module then $\tau_B^r \tau_{D'}^m N = \tau_{D'}^{m+r} N$ for $m \gg 0$ and $r \geq 0$. We have $\operatorname{id}_B \mathscr{H}(T) \leq 1$.

**Lemma.** *We have $\tau_{\mathscr{H}}^i X \in \mathscr{F}(T)$ for all $i \leq 1$.*

**Lemma.** *If $M \in \mathscr{I}(D')$ then $\operatorname{Hom}(M, \tau_{\mathscr{H}}^{-i} X) = 0$ for all $i \geq 0$.*

**Lemma.** *Let $X' := \operatorname{Ext}(T, X)$. Then $\operatorname{pd}_B X' \leq 1$, $\operatorname{pd}_B \tau_B X' \leq 1$ and $\tau_B X'$ is a simple $B$-module.*

We construct a titling $B$-module $\tilde{T}$ by the formula $\tilde{T} = \operatorname{Hom}(T, T_0) \oplus X'$. We have $\mathscr{F}(\tilde{T}) = \{M \in \operatorname{mod} B \mid \operatorname{Hom}(\tilde{T}, M) = 0\} = \operatorname{add}(\tau_B X')$. Let $\tilde{T}_0 = \operatorname{Hom}(T, T_0)$. Then

$$\operatorname{End}(\tilde{T}) = \begin{pmatrix} \operatorname{End}(\tilde{T}_0) & \operatorname{Hom}(\tilde{T}_0, X') \\ 0 & k \end{pmatrix}.$$

We have $\operatorname{End}(\tilde{T}_0) = \operatorname{End}(\tau_C^{r-1} DC) \simeq C$. Moreover

$$\begin{aligned} \operatorname{Hom}(\tilde{T}_0, X') &= \operatorname{Hom}(\operatorname{Hom}(T, T_0), \operatorname{Ext}(T, X)) \\ &\simeq \operatorname{Ext}(T_0, X) \simeq \operatorname{Ext}(T_0, Z) = \operatorname{Ext}(DC, \tau^{-r+1} Z) \simeq \tau_A^{-r} Z. \end{aligned}$$

We want to describe the preinjective component $\mathscr{I}$ of $\Gamma(C[\tau^{-r} Z])$ if $r \gg 0$. We have a natural division of the vertices of the quiver $Q$ into three classes, which consist of vertices such that corresponding simple modules are preprojective, regular or preinjective, respectively.

Let $S$ be a simple regular or preinjective. Given $m > 0$ there exists $r_0$ such that $\dim \operatorname{Hom}(\tau_C^{-r} Z, S) \geq m$ for all $r \geq r_0$. Since $\operatorname{Hom}(M, \mathscr{P}) = 0$ we have no arrows from $\omega$ to vertices corresponding to preprojective simple $C$-modules and we have many arrow to all the remaining vertices. Hence $\mathscr{I}$ contain $S_\omega$ and all injective modules which correspond to vertices which are regular or preinjective.