Introduction to quantum groups
and crystal bases

based on the talk by Markus Reineke (Wuppertal)

January 15, 2002

Let \( g \) be a finite dimensional complex Lie algebra. Examples of Lie algebras are \( \mathfrak{gl}_n \) and \( \mathfrak{sl}_n \), \( n \geq 2 \). We will always assume that \( g \) is a semisimple Lie algebra, i.e. \( g = \bigoplus_{i=1}^{k} g_i \), where \( g_i, i = 1, \ldots, k \), is a simple Lie algebra, that is \([-,-] \neq 0 \) and for each \( I \subset g \) such that \([g,I] \subset I\) we have either \( I = 0 \) or \( I = g_i \). The semisimple Lie algebras are classified by Dynkin diagrams or, equivalently, by Cartan matrices. For example, \( \mathfrak{sl}_n \) is a simple Lie algebra of type \( A_{n-1} \) and \( \mathfrak{sl}_3 \) corresponds to the matrix \( (2 - 1 - 1 2) \).

The representation of a Lie algebra \( g \) in a vector space \( V \) is a Lie algebra homomorphism \( g \to \text{gl}(V) \). Weyl showed that if \( g \) is semisimple then the category \( \text{mod} g \) of finite dimensional representations of \( g \) is semisimple. Let \( U(g) \) be the universal enveloping algebra of \( g \). The categories \( \text{mod} g \) and \( \text{mod} U(g) \) are equivalent, thus the category \( \text{mod} U(g) \) is semisimple.

Recall that a complex Lie algebra \( g \) has a decomposition \( g = n^- \oplus h \oplus n^+ \). For example if \( g = \mathfrak{sl}_n \), then \( n^- \) consists of the lower triangular matrices, \( h \) consists of the diagonal matrices and \( n^+ \) consists of the upper triangular matrices. We have the generators \( F_i \) of \( n^- \), \( H_i \) of \( h \), \( E_i \) for \( n^+ \), \( i \in I \), where \( I \) is the set of vertices of the corresponding Dynkin diagram. If \( g = \mathfrak{sl}_n \) then \( F_i = e_{i+1,i} \), \( H_i = e_{i,i} - e_{i+1,i+1} \) and \( E_i = e_{i,i-1} \), \( i = 1, \ldots, n-1 \). As the consequence \( U(n^+) \) is generated by \( E_i, i \in I \), as an algebra.

Let \( \lambda = (\lambda_i)_{i \in I} \in \mathbb{N}I, I_\lambda := \sum_{i \in I} U(n^+)E_i^{\lambda_i+1} \) and \( L_\lambda := U(n^+)/I_\lambda \). We define the action of \( U(g) \) on \( L_\lambda \) via \( F_i \mathcal{T} = 0 \) and \( H_i \mathcal{T} = \lambda_i \mathcal{T} \), \( i \in I \). It follows that \( L_\lambda, \lambda \in \mathbb{N}I \), form the complete set of simple \( U(g) \)-modules.

An interesting problem connected with the above description is the question about \( \dim L_\lambda \). Another one is the description of the restriction of \( L_\lambda \) to \( U(h) = \mathbb{C}[H_i \mid i \in I] \). This is answered by Weyl character formula, which says that \( \text{ch} L_\lambda := \sum_{\mu \in \mathbb{N}I} \dim(L_\lambda)_{\mu} e^\mu = \sum_{w \in \mathcal{W}} \text{sgn}(w)e^{w(\lambda + \rho)} \). However, there is still a question whether there is a “combinatorial formula” for \( \text{ch} L_\lambda \), i.e.
a formula of the form \((\dim L_\lambda)_\mu\) equals the number of certain combinatorial objects.

We know that \(L_\lambda \otimes L_\mu = \bigoplus_{\nu \in N I} L_\nu^{\lambda \mu}\) for some \(c_\lambda^\mu\). We may ask how to compute \(c_\lambda^\mu\). For type \(A\) the answer is contained in the Littlewood–Richardson rule.

We want to deform \(U(\mathfrak{g})\). However, complex semisimple Lie algebras are rigid, that is all deformations are trivial. Consequently, \(U(\mathfrak{g})\) is rigid as a cocommutative Hopf algebra. Happily, \(U(\mathfrak{g})\) is not rigid as a non-cocommutative Hopf algebra. From now on we will assume that \(\mathfrak{g}\) is of one of the types \(\mathbb{A}, \mathbb{D}\) or \(\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\).

**Theorem** (Serre). We have \(U(\mathfrak{n}^+) = \mathbb{C}\langle E_i \mid E_i \in I \rangle/[[E_i, E_j]] = 0\) if \(a_{ij} = 0\), and \([E_i, [E_i, E_j]] = 0\) if \(a_{ij} = -1\).

We have \([E_i, [E_i, E_j]] = E_i^2 E_j - 2E_i E_j E_i + E_j E_i^2\). Thus we may define \(\mathcal{H}(\mathfrak{n}^+) := \mathbb{C}\langle v \rangle\langle E_i \mid i \in I \rangle/[[E_i, E_j]] = 0\) if \(a_{ij} = 0\) and \(E_i^2 E_j = (v + v^{-1})E_i E_j E_i + E_j E_i^2 = 0\) if \(a_{ij} = -1\) and \(U_v(\mathfrak{n}^+)\) is the \(\mathbb{Z}[v, v^{-1}]\)-subalgebra of \(\mathcal{H}(\mathfrak{n}^+)\) generated by \(E_i^{(n)}\), \(i \in I\), \(n \in \mathbb{N}\), where \(E_i^{(n)} := \frac{1}{[n]}E_i^n\), and \([n] := \frac{v^n - v^{-n}}{v - v^{-1}}\). It follows easily that \(\mathbb{C}_1 \otimes \mathbb{Z}[v, v^{-1}] U_v(\mathfrak{n}^+) \simeq U(\mathfrak{n}^+)\), where \(\mathbb{C}_\mu\) denotes a 1-dimensional \(\mathbb{Z}[v, v^{-1}]\)-module with \(v\) acting by multiplication by \(\mu\).

Let \(Q\) be a quiver obtained from the diagram determining \(\mathfrak{g}\). For \(d \in NI\) we define \(R_d := \bigoplus_{\alpha:i \to j} \text{Hom}_k(k^{d_i}, k^{d_j})\) and \(G_d := \prod_{i \in I} \text{GL}(k^{d_i})\), where \(k = F_q\) for some \(q\). Then \(G_d\) acts on \(R_d\) via \((g_i) \ast (X_{\alpha}) := (g_j X_{\alpha} g_i^{-1})\). We put \(\mathcal{H}(Q) := \bigoplus_{d \in NI} \mathbb{C}^{G_d}(R_d)\), where \(\mathbb{C}^{G_d}(R_d)\) denotes the space of \(G_d\)-invariant complex functions on \(R_d\). The formula \((f \ast g)(X) := q^a \sum_{U \subset X} g(U) f(X/U)\) defines in \(\mathcal{H}(Q)\) a structure of an associative \(\mathbb{C}\)-algebra called the Hall algebra. We have \(\mathcal{H}(Q) \simeq \mathbb{C}^{\pi} \otimes \mathbb{Z}[v, v^{-1}] U_v(\mathfrak{n}^+)\).

Let \(\mathcal{B}_q(Q)\) be the set of the characteristic functions of all orbits in all \(R_d\). Then \(\mathcal{B}_q(Q)\) is a basis of \(\mathcal{H}(Q)\). There exists a basis \(\mathcal{B}(Q)\) of \(U_v(\mathfrak{n}^+)\), which specializes to \(\mathcal{B}_q(Q)\) for each \(q\). However, for different orientations \(Q\) of the diagram determining \(\mathfrak{g}\) the bases \(\mathcal{B}(Q)\) are different. Let \(\mathcal{L}(Q) := \mathbb{Z}[v^{-1}]\mathcal{B}(Q)\). If \(Q\) and \(Q'\) have the same underlying graph. Thus we put \(\mathcal{L} := \mathcal{L}(Q)\). If \(\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}\) is the canonical projection, then \(\pi(\mathcal{B}(Q)) = \pi(\mathcal{B}(Q'))\). We call \(B := \pi(\mathcal{B}(Q))\) the crystal basis of \(\mathcal{L}/v^{-1}\mathcal{L}\).

There exists the unique basis \(\mathcal{B}\) of \(U_v(\mathfrak{n}^+)\) such that \(\mathcal{B} \subset \mathcal{L}\), \(\pi(\mathcal{B}) = B\) and \(\overline{b} = b\) for all \(b \in \mathcal{B}\), where \(\overline{E}_i = E_i\) and \(\overline{v} := v^{-1}\). Th proof of the above fact uses degenerations.

Let \(\mathcal{B}_\mu\) be the specialization of \(\mathcal{B}\) to \(\mathbb{C}_\mu \otimes \mathbb{Z}[v, v^{-1}] \mathcal{B}\) and \(\pi_\lambda : U(\mathfrak{n}^+) \to L_\lambda\) be the canonical projection.
Theorem (Lusztig/Kashiwara). We have that $\pi_\lambda(\mathcal{B}_1) \setminus \{0\}$ is a basis of $L_\lambda$ for all $\lambda \in \mathbb{N}I$.

Proof. Fix $i \in I$ and choose an orientation $Q$ such that $i$ is a source in $Q$. Then, it follows that $\mathcal{B}_1(Q) \cap U(n^+)E_i^{\lambda_i+1}$ is a basis of $U(n^+)E_i^{\lambda_i+1}$. As the consequence $\mathcal{B}_1 \cap U(n^+)E_i^{\lambda_i+1}$ is a basis of $U(n^+)E_i^{\lambda_i+1}$ for all $i$. Hence $\mathcal{B}_1 \cap I_\lambda$ is a basis of $I_\lambda$ and the claim follows.

For example we have a basis of $\mathfrak{sl}_{n+1}$, which is parameterized by triangles $(a_{ij})_{1 \leq i \leq j \leq n}$, $a_{ij} \in \mathbb{N}$. The corresponding basis of $L_\lambda$ is parameterized by those $(a_{ij})$, which satisfy $\sum_{1 \leq k \leq i} a_{kj} - \sum_{1 \leq k < i} a_{k,j-1} \leq \lambda_j$ for $i \leq j$.