The Hall algebra as a quantum group

based on a talk by Markus Reineke (Wuppertal)

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Let Q be a Dynkin quiver and I the set of vertices of Q. We denote by $\langle -, - \rangle$ the Euler form of Q. Recall that the symmetrization of $\langle -, - \rangle$ is given by the Cartan matrix of Q.

For a finite field k with q elements we define $R_d := \bigoplus_{i \to j} \mathrm{Hom}_k(k^{d_i}, k^{d_j})$ and $G_d := \prod_{i \in I} \mathrm{GL}(k^{d_i})$. Recall that $\mathscr{H}_v(Q) := \bigoplus_{d \in \mathbb{N}I} \mathbb{C}^{G_d} R_d$ is an algebra with the convolution product

$$(f * g)(X) := v^{\langle d,e \rangle} \sum_{U \subset X} g(U) f(X/U),$$

where $f : R_d \to \mathbb{C}$, $g : R_e \to \mathbb{C}$ and $v^2 = q$. It follows from the definition that

$$(f_1 * \cdots * f_n)(X) = v^{\sum_{i<j} \langle d_i, d_j \rangle} \sum_{X = X_0 \supset X_1 \supset \cdots \supset X_n = 0} f_1(X_0/X_1) \cdots f_n(X_{n-1}/X_n),$$

where $f_i : R_i \to \mathbb{C}$.

The orbits of the action of G_d in R_d are in one to one correspondence with isoclasses of k-representation of Q of dimension vector d. Thus in $\mathscr{H}_v(Q)$ we have a basis $E_M := v^{\dim \mathrm{End}\, M - \dim M} \chi_{\mathscr{O}_M}$. We have

$$E_M E_N = v^{\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle} \sum_{[X]} v^{\dim \mathrm{End}\, M + \dim \mathrm{End}\, N - \dim \mathrm{End}\, X} F_{MN}^X(v^2) E_X,$$

where $F_{MN}^X(v^2)$ is the number of subrepresentations U of X such that $U \simeq N$ and $X/U \simeq M$.

Let R^+ be the set of positive roots of Q. By the theorem of Gabriel R^+ parameterizes the isoclasses of indecomposable representations of Q. For $\alpha \in R^+$ we denote by U_α the indecomposable representation of Q of dimension vector α. All isoclasses of representations of Q are parameterized by functions $R^+ \to \mathbb{N}$, where $f \mapsto \bigoplus_{\alpha \in R^+} U_\alpha^{f(\alpha)}$. Thus roughly speaking, we can deal with representations of Q over all fields at the same time.

Proposition (Ringel)**.** *We have* $F_{MN}^X \in \mathbb{Z}[v^2]$*.*

Let $\mathscr{H}(Q) := \bigoplus_{[M]} \mathbb{C}(v)E_M$ and $H(Q) := \bigoplus_{[M]} \mathbb{Z}[v,v^{-1}]E_M$, with multiplication

$$E_M E_N := v^{\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle} \sum_{[X]} v^{\dim \mathrm{End}(M) + \dim \mathrm{End}\, N - \dim \mathrm{End}\, X} F_{MN}^X(v^2) E_X.$$

Lemma 1. *If* U *is indecomposable then* $E_U^m = [m]! E_{U^m}$*.*

Proof. It follows easily by induction on m. □

Lemma 2. *If* $\mathrm{Hom}(N,M) = 0 = \mathrm{Ext}^1(M,N)$ *then* $E_M E_N = E_{M\oplus N}$*.*

Proof. It follows immediately by applying definition. □

Recall that the path algebra of Q is representation directed, hence there exists enumeration $U_1, \ldots, U_\nu$ of indecomposables representation of Q, such that $\mathrm{Hom}(U_j,U_i) = 0$ and $\mathrm{Ext}^1(U_i,U_j) = 0$ for $i < j$. Hence, if $M = \bigoplus_{i=1}^{\nu} U_i^{m_i}$, then $E_M = E_{U_1^{m_1}} \cdots E_{U_\nu^{m_\nu}}$.

We choose an order on $I = \{1, \ldots, n\}$, such that if there is an arrow $i \to j$ then $i < j$. Let E_i be the simple representation of Q corresponding to vertex i.

Lemma 3. *Let* $d = (d_1, \ldots, d_n) \in \mathbb{N}I$*. Then we have*

$$E_{E_1^{d_1}} \cdots E_{E_m^{d_m}} = \sum_{[M], \mathbf{dim}\, M = d} v^{-\dim \mathrm{Ext}^1(M,M)} E_M.$$

Proof. Any representation M of dimension vector d has a unique filtration $M = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = 0$ such that $M_{i-1}/M_i \simeq E_i^{d_i}$. Thus $E_{E_1^{d_1}} \cdots E_{E_m^{d_m}} = v^{\sum_{i<j} \langle d_i e_i, d_j e_j\rangle} \sum_{[M], \mathbf{dim}\, M = d} v^{\dim_i d_i^2 - \dim \mathrm{End}\, M} E_M$, and the claim follows. □

Let $\mathscr{U}_v(\mathfrak{n}^+) := \mathbb{C}(v)\langle E_i \mid i \in I\rangle / \mathscr{I}_v$, where $\mathscr{I}_v$ is the ideal in $\mathbb{C}(v)\langle E_i \mid i \in I\rangle$ generated by all elements $[E_i, E_j] = 0$, $i,j \in I$ such that there is no edge from i to j, and $E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_i E_j^2$, $i,j \in I$ such that there is an edge from i to j. We denote by $U_v(\mathfrak{n}^+)$ the $\mathbb{Z}[v,v^{-1}]$ subalgebra of $\mathscr{U}_v(\mathfrak{n}^+)$ generated by $E_i^{(n)} := \frac{1}{[n]!} E_i^n$, $i \in I$, $n \in \mathbb{N}$.

Theorem (Ringel)**.** *We have* $\mathscr{H}(Q) \simeq \mathscr{U}_v(\mathfrak{n}^+)$ *and* $H(Q) \simeq U_v(\mathfrak{n}^+)$*.*

Proof. By direct calculation it follows that there exists an algebra homomorphism $\eta : \mathscr{U}_v(\mathfrak{n}^+) \to \mathscr{H}(Q)$ such that $\eta(E_i) := E_{E_i}$. We have that $\eta(U_v(\mathfrak{n}^+)) \subset H(Q)$ and $\eta(E_i^{(n)}) \mapsto E_{E_i^n}$. We have to show that $H(Q)$ is generated by E_i^n, thus prove that each E_M belongs to span $E_{E_i^n}$.

If $\dim M = 1$ then M is simple and the claim is obvious. If $\dim M > 1$ then by Lemma 2 we may assume that $M \simeq U^q$, where U is indecomposable. By Lemma 3 $E_{U^q} = E_{E_1^{d_1}} \cdots E_{E_n^{d_n}} - \sum_{\mathbf{dim}\, N = d, N \not\simeq U^q} v^{-\dim \operatorname{Ext}^1(N,N)} E_N$. It follows that each N appearing in the sum is a direct sum if indecomposable representations V such that $\dim V < \dim U$, hence we may use induction.

Finally, we show that η is a monomorphism. Note that $\mathscr{U}_v(\mathfrak{n}^+)$ is $\mathbb{N}I$-graded by $\deg E_i = e_i$. Similarly, $\mathscr{H}(Q)$ is $\mathbb{N}I$-graded by $\deg E_M = \mathbf{dim}\, M$. Note that $\dim \mathscr{H}(Q)_d$ is the number of isoclasses of representation of dimension vector d, thus the number of functions $f : R^+ \to \mathbb{N}$ such that $\sum_\alpha f(\alpha)\alpha = d$. On the hand $\dim_{\mathbb{C}(v)} \mathscr{U}_v(\mathfrak{n}^+)_d = \dim_{\mathbb{C}} \mathscr{U}(\mathfrak{n}^+)$, and the latter equals the number of functions $f : R^+ \to \mathbb{N}$ such that $\sum_\alpha f(\alpha)\alpha = d$ by Poincere–Birkhoff–Witt theorem. $\square$