Generic extensions of (quiver) representations

based on the talk by Markus Reineke (Wuppertal)

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Let k be an algebraically closed field and Q a Dynkin quiver. We denote by I the set of vertices of Q.

Lemma. *Let M and N be representations of Q. There exists a unique representation $M * N$ such that $M * N \leq_{\deg} X$ if and only if there exist a short exact sequence $0 \to N' \to X \to M' \to 0$ with $M \leq_{\deg} M'$ and $N \leq_{\deg} N'$.*

Lemma. *If L, M and N are representations of Q then $(L * M) * N \simeq L * (M * N)$.*

Proof. We have the following commutative diagram with exact rows and columns

$$
\begin{array}{cccccccccc}
 & & & & & \xi_2 & & 0 & & \\
 & & & & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \\
\xi_1 & 0 & \to & N & \to & X & \to & M & \to & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \\
 & 0 & \to & N & \to & (L * M) * N & \to & L * M & \to & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & L & = & L & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}.
$$

From ξ_1 it follows that $M * N \leq_{\deg} X$, hence using ξ_2 we get $L * (M * N) \leq_{\deg} (L * M) * N$. Similarly, we show $(L * M) * N \leq_{\deg} L * (M * N)$, and the claim follows. □

We define $\mathscr{M}$ to be the set of all isoclasses of representations of Q. If we define multiplication in $\mathscr{M}$ by $[M] * [N] := [M * N]$, then we obtain in $\mathscr{M}$ a structure of monoid, called the monoid of generic extensions of Q.

Theorem. *We have* $\mathscr{M} \simeq \langle I\rangle/(ij = ji,$ *if there is no edge between* i *and* j, $iji = iij$ *and* $jij = ijj$, *if there is an arrow* $i \to j$). *The isomorphism is given by the assignment* $E_i \mapsto i$.

Recall that the Serre relation in $\mathscr{U}(\mathfrak{n}^+)$ is $E_iE_j - 2E_iE_jE_i + E_jE_i = 0$. The above relation can be "quantized" as follows $E_i^2E_j - (q+1)E_iE_jE_i + qE_jE_i^2 = 0$. For $q = 0$ we get $E_i^2E_j = E_iE_jE_i$. The above "quantized" relation appears in the non-twisted Hall algebra, that is in a Hall algebra with multiplication defined as $u_Mu_N := \sum_X F^X_{MN}(q)u_X$.

Let $U_1, \ldots, U_\nu$ be a list of indecomposable representations of Q such that $\operatorname{Ext}^1(U_i, U_j) = 0$ for $i \leq j$. If $M = \bigoplus_{i=1}^{\nu} U_i^{m_i}$ then $[M] = [U_1]^{*m_1} * \cdots * [U_\nu]^{*m_\nu}$. We have also the following result by Bongartz. Assume $\operatorname{Ext}^1(U \oplus V, U \oplus V) \simeq \operatorname{Ext}^1(U, V)$. There exists an exact sequence $0 \to V \to Y \to U \to 0$ if and only if $Y \leq_{\deg} U \oplus V$. In particular, $U * V$ has no selfextensions. Indeed, since $\operatorname{Ext}^1(X, X) = 0$ for each indecomposable representation X of Q, we only need to show $\operatorname{Ext}^1(X_i, \bigoplus_{j\neq i} X_j) = 0$ for each i, where $U * V = \bigoplus_i X_i$ is a decomposition of $U * V$ into a direct sum of indecomposable representations. Let

$$0 \to \bigoplus_{j\neq i} X_j \to Y \to X_i \to 0 \tag{$*$}$$

be an exact sequence. Then $Y \leq_{\deg} U * V \leq_{\deg} U \oplus V$. By the result of Bongartz, there exists an exact sequence $0 \to V \to Y \to U \to 0$, hence $U * V \leq_{\deg} Y$, and consequently $Y \simeq U * V$, thus the sequence $(*)$ splits. Using the above observations we may formulate the following algorithm for calculation of $M * N$.

Let $M = \bigoplus U_i^{m_i}$ and $N = \bigoplus U_i^{n_i}$. Then $[M] * [N] = [U_1]^{*m_1} \cdots [U_\nu]^{*m_\nu} * [U_1]^{*n_1} * \cdots * [U_\nu]^{*n_\nu}$. If $i < j$ then $[U_j] * [U_i] = [U_i]^{*a_i} * \cdots * [U_j]^{*a_j}$ for some $a_i, \ldots, a_j$ which can be read of from the Auslander–Reiten-quiver of Q. Repeated application of this rule brings $[M] * [N]$ to the form $[U_1]^{*x_1} * \cdots * [U_\nu]^{*x_\nu}$. Then $M * N \simeq \bigoplus_{i=1}^{\nu} U_i^{x_i}$.

Lemma. *Let* U *be an indecomposable representation of* Q. *If* M *is a representation of* Q *and* $0 \to M \to X \to U^n \to 0$ *is a universal extension, where* $n := \dim \operatorname{Ext}^1(U, M)$, *then* $X \simeq U^n * M$. *Moreover, if* $\operatorname{Ext}^1(M, U) = 0$ *then* $[U]^{*n} * [M] * [U] = [U]^{*(n+1)} * [M]$.

Let A be an algebra of finite representation type. We may defined generic extensions in the following way. If M and N are A-modules then there exists a unique module $M * N$ which is an extensions of M by N and $\dim \operatorname{End}(M \oplus N)$ has a minimal dimension among all extension of M by N. However, in general we do not have an equality $(L * M) * N = L * (M * N)$ is this case. Thus we may consider $\mathscr{M}(A)$ as the free associative monoid generated by all isoclasses of A-modules modulo the ideal generated by all elements of the form $[M][N]-[M*N]$. For example, if A is the path algebra of the bounded quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3,\ \beta\alpha = 0,$$

then $\mathscr{M}(A) \simeq \langle 1,2,3\rangle/(13 = 31, 112 = 121, 212 = 122, 312 = 123, 123 = 231, 223 = 232, 323 = 233)$. There is a question what is a connection between $\mathscr{M}(A)$ and the representation theory of A.

Let $A = k[[T]]$, thus in other words we consider the nilpotent representations of a one-loop quiver. It is known that the representations of A are parameterized by partitions. The theory of geometry of nilpotent orbits gives $M_\lambda * M_\mu = M_{\lambda+\mu}$, thus $(\mathcal{A})$ is a free commutative monoind in $[M_{(1,\dots,1)}]$.

The monoid of generic extensions of nilpotent representations of $\widetilde{\mathbb{A}}_n$ has been studied by Deng and Du (??). There is also a question what happens for poset representations.

Note, that we may treat the Dynkin quivers of types $\mathbb{B}$, $\mathbb{C}$, $\mathbb{F}$ and $\mathbb{G}$, as the corresponding Dynkin quivers of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$ with automorphism. For example, the quiver of type $\mathbb{B}_n$ may be viewed as the quiver of type $\mathbb{D}_{n+1}$ with automorphism identifying two vertices. Let (Q,γ) be a quiver with an automorphism. We define $\mathscr{M}(Q,\gamma)$ to be the submonoid of $\mathscr{M}(Q)$ consisting of γ-invariant representations of Q.

Let Q be a quiver of infinite representation type. We define $\mathscr{M}_d$ to be a family of all irreducible closed G_d stable subsets of R_d. In $\mathscr{M} = \bigcup_{d\in\mathbb{N}I}\mathscr{M}_d$, we may define $\mathscr{A} * \mathscr{B} := \{X \in R_{d+e} \mid \text{there exists a short sequence } 0 \to B \to X \to A \to 0, A \in \mathscr{A}, B \in \mathscr{B}\}$ for $\mathscr{A} \in \mathscr{M}_d$ and $\mathscr{B} \in \mathscr{M}_e$. We call $\mathscr{M}$ a monoid of families of representations of Q. We consider the submonoid $\mathscr{C}$ of $\mathscr{M}$ spanned by $R_i = \{E_i\}$, $i \in I$. If Q is Dynkin then $\mathscr{M} = \mathscr{C}$ and $\mathscr{M}$ coincides with the previous definition.

Theorem. *$\mathscr{C}$ is a quotient of $\langle I\rangle/(i^{n+1}j = i^n ji, ij^{n+1} = ijj^n$ if there is no arrow from j to i and there is n arrows form i to $j)$.*

Let $\omega = i_1\cdots i_\nu$ be a word in I and $\mathscr{E}_\omega := R_{i_1} * \cdots * R_{i_\nu}$. Then $\mathscr{E}_\omega$ is the set of all modules having composition series of type ω, i.e. $M \in \mathscr{E}_\omega$ if and only if $M = M_0 \supset M_1 \supset \cdots \supset M_\nu = 0$ with $M_{k-1}/M_k \simeq E_k$. The answer to the question when $\mathscr{E}_\omega = \mathscr{E}_{\omega'}$ is encoded in $\mathscr{C}$. Applying Schofield's theory of "general properties of representations" we also have that $R_d * R_e = R_{d+e}$ if and only if $\langle e', d\rangle \geq 0$ whenever $R_{e'} * R_{e-e'} = R_e$.