Automorphisms of repetitive algebras

based on the talk by Kunio Yamagata (Tokyo)

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Let K be a field and A a finite dimensional K-algebra. By $\hat{A}$ we will denote the repetitive algebra of A. An automorphism φ of $\hat{A}$ will be called rigid provided $\varphi(A_m) \subset A_m$ for all m. The set of the rigid automorphisms of $\hat{A}$ will be denoted by $\mathrm{Rig}(\hat{A})$. For an invertible element a of A we will denote by θ_a the corresponding inner automorphism of A.

The pair $(\boldsymbol{\sigma}, \mathbf{u})$, $\boldsymbol{\sigma} \in \prod_{m\in\mathbb{Z}} \mathrm{Aut}(A)$, $\mathbf{u} \in \prod_{m\in\mathbb{Z}} A^*$, is called admissible if $\sigma_{m+1} = \theta_{u_m}\sigma_m$ for any m. For an admissible pair $(\boldsymbol{\sigma}, \mathbf{u})$ the map $\varphi_{\boldsymbol{\sigma},\mathbf{u}} : \hat{A} \to \hat{A}$ given by

$$\begin{aligned} \varphi_{\boldsymbol{\sigma},\mathbf{u}}(a) &:= \sigma_m(a),\ a \in A_m, \\ \varphi_{\boldsymbol{\sigma},\mathbf{u}}(f) &:= f\sigma_m^{-1}L_{u_m^{-1}},\ f \in DA_m, \end{aligned}$$

where $L_a : A \to A$ is the left multiplication by a, is a rigid automorphism of $\hat{A}$. Moreover, for a rigid automorphism φ of $\hat{A}$ there exists an admissible pair $(\boldsymbol{\sigma}, \mathbf{u})$ such that $\varphi = \varphi_{\boldsymbol{\sigma},\mathbf{u}}$. Finally, $\varphi_{\boldsymbol{\sigma},\mathbf{u}} = \varphi_{\boldsymbol{\tau},\mathbf{v}}$ if and only if $\boldsymbol{\sigma} = \boldsymbol{\tau}$ and $\mathbf{u} = \mathbf{v}$. For $\sigma \in \mathrm{Aut}(A)$ denote $\hat{\sigma} := (\sigma) \in \prod_{m\in\mathbb{Z}} \mathrm{Aut}(A)$. Similarly, for $u \in A^*$ let $\hat{u} := (u_i) \in \prod_{m\in\mathbb{Z}} A^*$. We have $\mathrm{Id}_{\hat{A}} = \varphi_{\hat{\mathrm{Id}}_A,\hat{1}}$, $\varphi_{\boldsymbol{\tau},\mathbf{v}}\varphi_{\boldsymbol{\sigma},\mathbf{u}} = \varphi_{\boldsymbol{\tau}\boldsymbol{\sigma},\mathbf{v}\boldsymbol{\tau}(\mathbf{u})}$ and $\varphi_{\boldsymbol{\sigma},\mathbf{u}}^{-1} = \varphi_{\boldsymbol{\sigma}^{-1},\boldsymbol{\sigma}^{-1}(\mathbf{u}^{-1})}$.

We define the group $\mathscr{S}(A^*) = (\mathscr{S}, \circ)$ in the following way. Let $\mathscr{S} := \prod_{m\in\mathbb{Z}} A^*$. For $\mathbf{u}$ and $\mathbf{v}$ in $\mathscr{S}$ we put $\mathbf{u} \circ \mathbf{v} := \mathbf{u}\theta(\mathbf{u})(\mathbf{v})$, where

$$\theta(\mathbf{u})_m := \begin{cases} \theta_{u_{m-1}\cdots u_0} & m > 0 \\ \mathrm{Id}_A & m = 0 \\ \theta^{-1}_{u_{-1}\cdots u_m} & m < 0 \end{cases}.$$

Note that for $\mathbf{u} \in \prod A^*$ the pair $(\theta(\mathbf{u}), \mathbf{u})$ is rigid. Moreover, if $(\boldsymbol{\sigma}, \mathbf{u})$ is a rigid pair and $\sigma_0 = \mathrm{Id}_A$, then $(\boldsymbol{\sigma}, \mathbf{u}) = (\theta(\mathbf{u}), \mathbf{u})$. Let $\Phi : \mathrm{Rig}(\hat{A}) \to \mathrm{Aut}(A)$ be a group homomorphism given by $\Phi(\varphi_{\boldsymbol{\sigma},\mathbf{u}}) := \sigma_0$ and $\Psi : \mathrm{Aut}(A) \to \mathrm{Rig}(\hat{A})$ be $\Psi(\sigma) := \varphi_{\hat{\sigma},\hat{1}}$. Then $\Phi\Psi = \mathrm{Id}_{\mathrm{Aut}(A)}$. Moreover, $\mathrm{Ker}\,\Phi = \{\varphi_{\theta(\mathbf{u}),\mathbf{u}} \mid \mathbf{u} \in \prod_{m\in\mathbb{Z}} A^*\}$ is isomorphic to $\mathscr{S}(A^*)$ as a group.

It follows from the above considerations that we have the split exact sequence $1 \to \mathscr{S}(A^*) \to \mathrm{Rig}(\hat{A}) \to \mathrm{Aut}(A) \to 1$. It is known that there exists a group homomorphism $\chi : \mathrm{Aut}(A) \to \mathrm{Aut}(\mathscr{S}(A^*))$ such that $\mathscr{S}(A^*) \rtimes_\chi \mathrm{Aut}(A) \simeq \mathrm{Rig}(\hat{A})$. We obtain by direct calculations that $[\chi(\sigma)](\mathbf{u}) = \hat{\sigma}(\mathbf{u})$. Note that if A^* is commutative, then $\mathrm{Rig}(\hat{A}) \simeq \prod_{m\in\mathbb{Z}} A^* \rtimes_\chi \mathrm{Aut}(A)$.

Let $\mathscr{E} = \{e_1, \ldots, e_n\}$ be the complete set of orthogonal primitive idempotents of A. We have a category $(A, \mathscr{E})$ with $\mathrm{Obj}(A, \mathscr{E}) = \mathscr{E}$. If $\hat{\mathscr{E}} := \coprod_{m\in\mathbb{Z}} \mathscr{E}$, then $(\hat{A}, \hat{\mathscr{E}})$ is a category with $\mathrm{Obj}(\hat{A}, \hat{\mathscr{E}}) = \hat{\mathscr{E}}$. The groups $\mathrm{Aut}(A, \mathscr{E})$ and $\mathrm{Rig}(\hat{A}, \mathscr{E})$ are subgroups of $\mathrm{Aut}(A)$ and $\mathrm{Rig}(\hat{A})$, respectively. If $u \in A^*$ then $\theta_u \in \mathrm{Aut}(A, \mathscr{E})$ if and only if $ue_i = e_i u$ for all i and if and only if $u \in \sum_{i=1}^n (e_iAe_i)^*$. Note that $\sum_{i=1}^n (e_iAe_i)^*$ is isomorphic as a group with $\prod_{i=1}^n (e_iAe_i)^*$. We also have $\mathrm{Rig}(\hat{A}, \hat{\mathscr{E}}) = \{\varphi_{\boldsymbol{\sigma},\mathbf{u}} \in \mathrm{Rig}(\hat{A}) \mid \theta_m \in \mathrm{Aut}(A, \mathscr{E})$ and $\sigma_m \in \mathrm{Aut}(A, \mathscr{E})\}$. Finally, we get $\mathrm{Ker}\,\Phi = \{\varphi_{\theta(\mathbf{u}),\mathbf{u}} \mid u \in \prod_{i=1}^m (e_iAe_i)^*\}$, hence $\mathrm{Rig}(\hat{A}, \hat{\mathscr{E}})) \simeq \mathscr{S}(\prod_{i=1}^n (e_iAe_i)^*) \rtimes_\chi \mathrm{Aut}(A, \mathscr{E})$. Since $(e_iAe_i)^*$ is commutative for e_iAe_i is uniserial, thus in this case $\mathrm{Rig}(\hat{A}, \hat{\mathscr{E}})) \simeq \prod_{m\in\mathbb{Z}} \prod_{i=1}^n (e_iAe_i)^* \rtimes_\chi \mathrm{Aut}(A, \mathscr{E})$. Similarly, if A is a triangular algebra then $\mathrm{Rig}(\hat{A}, \hat{\mathscr{E}}) \simeq \prod_{m\in\mathbb{Z}} \prod_{i=1}^n K \rtimes_\chi \mathrm{Aut}(A, \mathscr{E})$.

Let $g \in \mathrm{Rig}(\hat{A})$, $h \in \mathrm{Rig}(\hat{B})$. Recall that the category isomorphism $F : \hat{A} \to \hat{B}$ such that $Fg = hF$, induces a category isomorphism $\hat{A}/(g\nu^m) \simeq \hat{B}/(h\nu^m)$ for any m. Let $\varphi_{\boldsymbol{\sigma},\mathbf{u}}, \varphi_{\boldsymbol{\tau},\mathbf{v}} \in \mathrm{Rig}(\hat{A})$. It follows that $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are conjugate if and only if there exists an algebra isomorphism $\hat{A}/(\varphi_{\boldsymbol{\sigma},\mathbf{u}}\nu) \simeq \hat{A}/(\varphi_{\boldsymbol{\tau},\mathbf{v}}\nu)$ induced from the categorical isomorphism $\hat{A} \simeq \hat{A}$.