Koszul algebras

based on the talk by Dan Zacharia (Syracuse)

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Let $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ be a graded algebra such that $\dim \Lambda_i < \infty$ for each i and $\Lambda_0 = K \times \cdots \times K$. We put $J := \bigoplus_{i\geq 1} \Lambda_i$. Note that every graded simple Λ-module generated in degree 0 is isomorphic to a summand of Λ_0. The ext-algebra $E(\Lambda)$ of Λ is $E(\Lambda) := \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$. Note that $E(\Lambda)$ is a graded K-algebra with the multiplication given by the Yoneda product. We describe this multiplication explicitly.

Let M be a finitely generated graded Λ-module and fix a minimal graded resolution of M

$$\cdots \to P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to M \to 0.$$

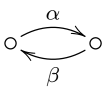
The minimality of the above resolution means that $\operatorname{Im} \delta_n \subset JP_{n-1}$. Consequently for each $n \geq 1$ we have $\operatorname{Hom}_\Lambda(\delta_n, \Lambda_0) = 0$, hence $\operatorname{Ext}^n_\Lambda(M, \Lambda_0) = \operatorname{Hom}_\Lambda(P_n, \Lambda_0)$ for $n \geq 0$.

Let $\xi \in \operatorname{Ext}^i_\Lambda(\Lambda_0, \Lambda_0)$ and $\mu \in \operatorname{Ext}^j_\Lambda(\Lambda_0, \Lambda_0)$, and

$$\cdots \to P_n \xrightarrow{\delta_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \to \Lambda_0 \to 0$$

be the minimal graded resolution of Λ_0. Denote by ξ and μ the corresponding maps $\xi : P_i \to M$ and $\mu : P_j \to M$. The map $\mu : P_j \to \Lambda_0$ induces maps $l_k : P_{j+k} \to P_k$, $k \geq 0$. It appears that $\xi\mu$ correspond to ξl_i.

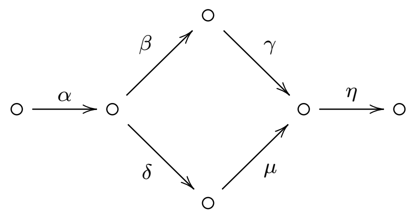
Usually $E(\Lambda)$ is very big. It need not be finitely generate even if Λ is finite dimensional over K. For example, if Λ is the path algebra of the quiver



bounded by $\alpha\beta\alpha$, then $E(\Lambda)$ is not finitely generated.

Let $\Lambda^!$ be the subalgebra of $E(\Lambda)$ generated by the degree 0 and 1 parts, that is $\Lambda^! = E(\Lambda)_0 \oplus E(\Lambda)_1 \oplus E(\Lambda)_1^2 \oplus \cdots$. We call $\Lambda^!$ the shriek algebra of Λ. An algebra Λ is called Koszul if and only if $E(\Lambda) = \Lambda^!$.

Let $\Lambda = KQ/I$, where Q is a finite quiver and I is a homogeneous admissible ideal (KQ is graded by the lengths of the paths). If Λ is a Koszul algebra then Λ is quadratic, that is I is generated by linear combinations of paths of length 2. The converse implication does not hold. For example, if Λ is the path algebra of the quiver



bounded by relations $\alpha\beta$, $\beta\gamma - \delta\mu$, $\mu\eta$, then $E(\Lambda)$ is not Koszul.

Let Q be a quiver. In the subspace V of KQ spanned by all paths of length 2 we introduce a bilinear form $\langle -, - \rangle : V \times V \to K$ given by $\langle p, q \rangle := \delta_{p,q}$ for paths p, q of length 2, where $\delta_{x,y}$ is the Kronecker delta. For $X \subset V$ we put $X^{\perp} := \{v \in V \mid \langle X, v \rangle =\}$.

Theorem. *Let $\Lambda = KQ/I$ be a quadratic algebra. Then $\Lambda^! = KQ/\langle I_2^{\perp} \rangle$, where $I_2 := I \cap \Lambda_2$.*

Using the above theorem we obtain that for $\Lambda := K[X_1, \dots, X_n] = K\langle X_1, \dots, X_n \rangle / \langle X_i X_j - X_j X_i \rangle$, the shriek algebra $\Lambda^!$ is the exterior algebra in n variables, that is $\Lambda^! = K\langle X_1, \dots, X_n \rangle / \langle X_i^2, X_i X_j + X_j X_i \rangle$. Similarly, if $\Lambda = KQ$ then $\Lambda^! = KQ/J^2$.

Let $M \in \operatorname{gr} \Lambda$ and assume $M = \bigoplus_{i \geq j} M_i$. We say that M has a linear resolution (M is a Koszul module) if there exists a graded resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M, such that P_k is generated in degree $k + j$ for each k. In particular, if M is a Koszul module then M is generated in degree j.

Theorem. *An algebra Λ is Koszul if and only if Λ_0 is a Koszul module.*

We have the following examples of Koszul algebras.

(1) The polynomial algebra and the exterior algebra are Koszul.

(2) Hereditary algebras are Koszul.

(3) If I is generated by quadratic monomials then KQ/I is a Koszul algebra.

(4) If I is an ideal in $K[X_1, \dots, X_n]$ generated by a regular sequence of quadratic forms then $K[X_1, \dots, X_n]/I$ is a Koszul algebra.

(5) Let Δ be a finite simplicial complex in vertices $v_1, \ldots, v_n$. We define the Stanley–Reisner ring $K[\Delta]$ of Δ as $K[\Delta] = K[X_1, \ldots, X_n]/I_\Delta$, where I_Δ is generated by $X_{i_1} \cdots X_{i_t}$ such that $\{v_{i_1} \cdots v_{i_t}\} \notin \Delta$. If Δ is a baricentric subdivision then $K[\Delta]$ is Koszul.

(6) If Λ is Koszul then Λ^{op} is Koszul.

(7) If Λ and Γ are Koszul then $\Lambda \otimes_k \Gamma$ is Koszul. In particular, $\Lambda^e = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ is Koszul.

Assume for the moment $\Lambda = K$. The Hilbert series of Λ is by definition $H_\Lambda := \sum_{i\geq 0} \dim \Lambda_i t^i$. In a similar way we can define the Hilbert series H_M of $M \in \operatorname{gr}\Lambda$. We can also define the Poincar series of $M \in \operatorname{gr}\Lambda$ as $P_\Lambda^M := \sum_{i\geq 0} \dim \operatorname{Ext}^i_\Lambda(M, K) t^i$. If Λ is a Koszul algebra and M is a Koszul module generated in degree 0, then $P_\Lambda^M(t) = \frac{H_M(-t)}{H_\Lambda(-t)}$. In particular, if $P_\Lambda^K(t) H_\Lambda(-t) = 1$. Note that $P_\Lambda^K = H_{E(\Lambda)}$.

Theorem. *If Λ is a quadratic algebra with $\Lambda_0 = K$ then the following conditions are equivalent.*

(1) Λ *is Koszul.*

(2) $H_{E(\Lambda)}(t) H_\Lambda(-t) = 1$.

(3) $P_\Lambda^K = H_{\Lambda^!}$.

Roos observed that in general the equality $H_{\Lambda^!}(t) H_\Lambda(-t) = 1$ does not imply that Λ is a Koszul algebra.

If Λ is a Koszul algebra and M a Koszul Λ-module then P_Λ^M is a rational function. Jacobsson showed that in general P_Λ^M need not be rational. However, Martinez-Villa and Zacharia showed that if Λ is a finite dimensional Koszul algebra such that $E(\Lambda)$ is noetherian of finite global dimension then P_Λ^M is rational for each $M \in \operatorname{gr}\Lambda$.

Let Λ be a graded algebra. We can define a contravariant functor $\mathscr{E} : \operatorname{mod}\Lambda \to \operatorname{Gr} E(\Lambda)$ given by $\mathscr{E}(M) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(M, \Lambda_0)$. Note that $\mathscr{E}(S)$ is a graded projective $E(\Lambda)$-module if S is graded simple. On the other hand, if P is an indecomposable graded projective Λ-module then $\mathscr{E}(P)$ is a graded simple $E(\Lambda)$-module.

Lemma. *If Λ is a Koszul algebra and M is a Koszul Λ-module then ΩM and JM are Koszul.*

Proposition. *Let* Λ *be a Koszul algebra and* M *a Koszul module. Then we have an exact sequence in* $\operatorname{gr} E(\Lambda)$

$$0 \to \mathscr{E}(JM)(-1) \to \mathscr{E}(M/JM) \to \mathscr{E}(M) \to 0,$$

that is $\Omega\mathscr{E}(M) = \mathscr{E}(JM)(-1)$.

Proof. Assume for simplicity that M is generated in degree 0. By applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & JM & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \Omega M & \to & P_0 & \to & M & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \Omega(M/JM) & \to & P_0 & \to & M/JM & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & N & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

we get $N \simeq JM$, hence the exact sequence

$$0 \to \Omega M \to \Omega(M/JM) \to JM \to 0.$$

Using that ΩM and JM are Koszul modules, calculating the minimal graded resolutions and using the rule for finding $\operatorname{Ext}^i_\Lambda(L, \Lambda_0)$, we get for each $i \geq 0$ the sequence

$$0 \to \operatorname{Ext}^i_\Lambda(JM, \Lambda_0) \to \operatorname{Ext}^i_\Lambda(\Omega(M/JM), \Lambda_0) \to \operatorname{Ext}^i_\Lambda(\Omega M, \Lambda_0) \to 0$$

and the claim follows, since $\operatorname{Ext}^i_\Lambda(\Omega(M/JM), \Lambda_0) = \operatorname{Ext}^{i+1}_\Lambda(M/JM, \Lambda_0)$ and $\operatorname{Ext}^i_\Lambda(\Omega M, \Lambda_0) = \operatorname{Ext}^{i+1}_\Lambda(M, \Lambda_0)$. $\square$

For a Koszul algebra Λ we denote by K_Λ the subcategory in $\operatorname{gr} \Lambda$ of Koszul modules generated in degree 0.

Theorem. *Let* Λ *be a Koszul algebra. Then* $E(\Lambda)$ *is a Koszul algebra and* $\mathscr{E}(M) \in K_{E(\Lambda)}$ *for* $M \in K_\Lambda$.

Proof. Using the above proposition, we get for each $M \in K_\Lambda$ the following linear resolution

$$\cdots \to \mathscr{E}(JM/J^2M)(-2) \to \mathscr{E}(JM/J^2M)(-1) \to \mathscr{E}(M/JM) \to \mathscr{E}(M) \to 0$$

of $\mathscr{E}(M)$ over $E(\Lambda)$. Since $E(\Lambda)_0 = \mathscr{E}(\Lambda)$ it follows that $E(\Lambda)_0$ is a Koszul module thus $E(\Lambda)$ is Koszul algebra. $\square$

Theorem. *Let* Λ *be a Koszul algebra. There exist dualities* $\mathscr{E} : K_\Lambda \to K_{E(\Lambda)}$ *and* $\mathscr{F} : E_{E(\Lambda)} \to K_\Lambda$ *inverse to each other, given by* $\mathscr{E}(M) := \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(M, \Lambda_0)$ *and* $\mathscr{F}(X) := \bigoplus_{i\geq 0} \operatorname{Ext}^i_{E(\Lambda)}(X, E(\Lambda)_0)$.

Theorem. *The following conditions are equivalent for a graded algebra* Λ.

(1) Λ *is Koszul.*

(2) $E(E(\Lambda)) \simeq \Lambda$ *as graded algebras.*

(3) Λ *is a Koszul* Λ^e*-module.*

(4) *Quiver of* Λ *equals the quiver of* $E(\Lambda)$.