Generalizations of Koszul modules

based on the talk by Dan Zacharia (Syracuse)

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The talk is based on the papers by Green–Martinez-Villa and Martinez-Villa–Zacharia.

Let Λ be a Koszul algebra. A Λ-module M is called weakly Koszul if there exists a projective resolution

$$(*) \qquad \cdots \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

of M, such that $J^{k+1}P_n \cap \operatorname{Ker} f_n = J^k \operatorname{Ker} f_n$ for all k and n. Note, if M is a Koszul module then M is weakly Koszul. On the other hand, each weakly Koszul module generated in one degree is a Koszul module. Finally, if M is weakly Koszul then obviously ΩM is also weakly Koszul.

A short exact sequence $0 \to A \to B \to C \to 0$ is called a relative extension if $J^k A = J^k B \cap A$ for each k. Thus, we may say that M is weakly Koszul if and only if for a projective resolution $(*)$ of M, the sequence $0 \to \operatorname{Ker} f_n \to JP_n \to J \operatorname{Ker} f_{n-1} \to 0$ is a relative extension for each n.

Lemma. *A sequence* $0 \to A \to B \to C \to 0$ *is a relative extension if and only if for every* k *the sequence* $0 \to A/J^kA \to B/J^kB \to C/J^kC \to 0$ *is an exact sequence.*

Proof. Let $0 \to A \to B \to C \to 0$ be a relative extension. Using that $J^kA = A \cap J^kB$ we get for each k the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J^kA & \to & J^kB & \to & J^kC & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

with exact rows and vertical maps being monomorphism. By passing to cokernels we obtain the desired exact sequence.

The converse implication is obvious. $\square$

Proposition. *Let* $0 \to A \to B \to C \to 0$ *be a relative extension with* A *weakly Koszul. Then for each* n *the sequence*

$$0 \to \Omega^n A \to \Omega^n B \to \Omega^n C \to 0$$

is a relative extension.

Proof. It is enough to show the claim for $n = 1$, since with A also ΩA is weakly Koszul.

Let P_A, P_B and P_C be projective covers of A, B and C respectively. Since we have a short exact sequence

$$0 \to A/JA \to B/JB \to C/JC \to 0$$

we get $P_B \simeq P_A \oplus P_C$. Consequently, we get a short exact sequence

$$0 \to \Omega A \to \Omega B \to \Omega C \to 0.$$

Applying the functor $\Lambda/J^k \otimes_\Lambda -$ to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega A & \to & \Omega B & \to & \Omega C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & JP_A & \to & JP_B & \to & JP_C & \to & 0 \end{array}$$

with exact rows and vertical maps being monomorphisms, we get a commutative diagram

$$\begin{array}{ccccccccc} & & \Omega A/J^k\Omega A & \to & \Omega B/J^k\Omega B & \to & \Omega C/J^k\Omega C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & JP_A/J^{k+1}P_A & \to & JP_B/J^{k+1}P_B & \to & JP_C/J^{k+1}P_C & \to & 0 \end{array}$$

with exact rows. Since A is weakly Koszul, we get that the map $\Omega A/J^k\Omega A \to JP_A/J^{k+1}P_A$ is a monomorphism, thus also $\Omega A/J^k\Omega A \to \Omega B/J^k\Omega B$ is a monomorphism and it finishes the proof. □

As the consequence of the above proposition we get that for a relative extension $0 \to A \to B \to C \to 0$ with A weakly Koszul, we have $P_n^{(B)} = P_n^{(A)} \oplus P_n^{(C)}$, where $P^{(A)}$, $P^{(B)}$ and $P^{(C)}$ are minimal projective resolutions of A, B and C, respectively.

Proposition. (1) *Let* $0 \to A \to B \to C \to 0$ *be a relative extension. If* A *and* B *are weakly Koszul then also* C *is weakly Koszul.*

(2) *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence such that and* A *and* C *are weakly Koszul and* $JA = JB \cap A$. *Then the above sequence is a relative extensions and* B *is weakly Koszul.*

Proof. For the proof of the first part note that we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega A & \to & \Omega B & \to & \Omega C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & JP_A & \to & JP_B & \to & JP_C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & JA & \to & JB & \to & JC & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. The first two columns are relative extensions. We need to show that the third one is also a relative extension. After applying the functor $\Lambda/J^k \otimes_\Lambda -$ we get the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega A/J^k\Omega A & \to & \Omega B/J^k\Omega B & \to & \Omega C/J^k\Omega C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & JP_A/J^{k+1}P_A & \to & JP_B/J^{k+1}P_B & \to & JP_C/J^{k+1}P_C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & JA/J^{k+1}A & \to & JB/J^{k+1}B & \to & JC/J^{k+1}C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and two first columns. It follows that also the last column has to be exact and it finishes the proof. □

Corollary. *Let Λ be a Koszul algebra. If M is a weakly Koszul Λ-module then JM is also weakly Koszul.*

Proof. Recall that we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & JM & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \Omega M & \to & P_0 & \to & M & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \Omega(M/JM) & \to & P_0 & \to & M/JM & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & JM & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with exact rows and columns, where P_0 is the projective cover of M and M/JM. Since M and M/JM, and consequently ΩM and $\Omega(M/JM)$, are weakly Koszul, it is enough to show that this sequence is a relative extension and apply the previous result. The first part follows, since for each k we have the commutative diagram

$$\begin{array}{ccccc} 0 & \to & \Omega M/J^k\Omega M & \to & JP_0/J^{k+1}P_0 \\ & & \downarrow & & \| \\ 0 & \to & \Omega(M/JM)/J^k\Omega(M/JM) & \to & JP_0/J^{k+1}P_0 \end{array}$$

with exact rows, what implies that $\Omega M/J^k\Omega M \to \Omega(M/JM)/J^k\Omega(M/JM)$ has to be a monomorphism. $\square$

Let Λ be a finite dimensional module and $0 \to A \to B \xrightarrow{p} C \to 0$ an exact sequence with p an irreducible epimorphism, C indecomposable and A not simple. Then $JB \cap A = JA$. As the consequence of the above observation we get that if $0 \to A \to B \to C \to 0$ is an almost split sequence with A and C weakly Koszul and A not simple then B is weakly Koszul.

Let Λ be a Koszul algebra and M a graded module generated in degrees $i_0 < i_1 < \cdots < i_p$. We denote by K_M the submodule of M generated by the degree i_0 part of M, that is $K_M := \langle M_{i_0} \rangle = M_{i_0} \oplus \Lambda_1 M_{i_0} \oplus \Lambda_2 M_{i_0} \oplus \cdots$. If M is a weakly Koszul module, then K_M is a Koszul module and a sequence $0 \to K_M \to M \to L \to 0$ is a relative extension such that L is a weakly Koszul module generated in degrees $i_1 < i_2 < \cdots < i_p$.

Theorem. *Let Λ be a Koszul algebra. A finitely generated graded Λ-module M is weakly Koszul if and only if $G(M) := \bigoplus_{i\geq 0} J^iM/J^{i+1}M$ is Koszul.*

Proof. We only show that if M is weakly Koszul then $G(M)$ is Koszul, since the other implication is easy. For $p = 0$ the statement is trivial. If $p > 0$ we get that $0 \to G(K_M) \to G(M) \to G(L) \to 0$ is a relative extension, using that $0 \to K_M \to M \to L \to 0$ is a relative extension. Now the claim follows easily. $\square$

Theorem. *If Λ is a Koszul algebra then $M \in \operatorname{gr}\Lambda$ is weakly Koszul if and only if $\mathscr{E}(M) \in K_{E(\Lambda)}$.*

Proof. First we show that for a weakly Koszul module $M \in \operatorname{gr}\Lambda$ we have $\mathscr{E}(M) \in K_{E(\Lambda)}$. First note that for a weakly Koszul module N we have an exact sequence

$$0 \to \mathscr{E}(JN)(-1) \to \mathscr{E}(N/JN) \to \mathscr{E}(N) \to 0.$$

The proof of this fact uses that $0 \to \Omega N \to \Omega(N/JN) \to JN \to 0$ is a relative extension of weakly Koszul modules. Using the above sequence we can construct a linear resolution

$$\cdots \to \mathscr{E}(JM/J^2M)(-1) \to \mathscr{E}(M/JM) \to \mathscr{E}(M) \to 0$$

of $\mathscr{E}(M)$, thus $\mathscr{E}(M) \in K_{E(\Lambda)}$.

In the proof of the converse implication we will need the following lemma, which we will not prove.

Lemma. *If $\mathscr{E}(M) \in K_{E(\Lambda)}$ then for each k we have the exact sequence $0 \to \Omega^k M \to \Omega^k(M/JM) \to \Omega^{k-1}JM \to 0$ with $J\Omega^k(M/JM) \cap \Omega^k M = J\Omega^k M$.*

Observe that if $\mathscr{E}(M) \in K_{E(\Lambda)}$ then also $\mathscr{E}(J^k M) \in K_{E(\Lambda)}$ for each k. It is enough to show it for $k = 1$. Applying the above lemma we get the following exact sequence

$$0 \to \mathscr{E}(JM)(-1) \to \mathscr{E}(M/JM) \to \mathscr{E}(M) \to 0,$$

hence $\mathscr{E}(JM)(-1) = \Omega\mathscr{E}(M)$ is Koszul, thus $\mathscr{E}(JM) \in K_{E(\Lambda)}$.

Another necessary fact is the following.

Proposition. *Let Λ be a Koszul algebra and $0 \to A \to B \to C \to 0$ a short exact sequence with $JA = JB \cap A$ such that $\mathscr{E}(A)$, $\mathscr{E}(B)$, $\mathscr{E}(C)$ are all generated in degree 0. Then $J^2A = J^2B \cap A$.*

We want to show that $J^k\Omega M = J^k\Omega(M/JM) \cap \Omega M$ for each k. For $k = 1$ the claim holds by the above lemma. Let $k \geq 2$. From the above lemma we have a short exact sequence

$$0 \to J^{k-2}\Omega M \to J^{k-2}\Omega(M/JM) \to J^{k-1}M \to 0.$$

Denote $A = J^{k-2}\Omega M$, $B = J^{k-2}\Omega(M/JM)$ and $C = J^{k-1}M$. Then using inductive hypothesis we get $JB \cap A = J^{k-1}\Omega(M/JM) \cap J^{k-2}\Omega M = J^{k-1}\Omega(M/JM) \cap \Omega M \cap J^{k-2}\Omega M = J^{k-1}\Omega M \cap J^{k-2}\Omega M = J^{k-1}\Omega M = JA$. Moreover, $\mathscr{E}(C) = \mathscr{E}(J^k M) \in K_{E(\Lambda)}$ be the previous step. Similarly, $\Omega(M/JM)$ is Koszul being a syzygy of a Koszul module, hence $\mathscr{E}(B) \in K_{E(\Lambda)}$. Finally, we know $\mathscr{E}(\Omega M)(-1) = J\mathscr{E}(M)$ for $\mathscr{E}(M) \in K_{E(\Lambda)}$. Thus, $\mathscr{E}(\Omega M) \in K_{E(\Lambda)}$, hence also $\mathscr{E}(A) \in K_{E(\Lambda)}$. Using the above proposition, we can perform the inductive step.

Since $\mathscr{E}(\Omega M) \in K_{E(\Lambda)}$, thus in order to show that M is weakly Koszul it is enough now to show that $J^{k+1}P_0 \cap \Omega M = J^k\Omega M$ for each k, where P_0 is a projective cover of M. For $k = 0$ the claim is trivial. Let $k \geq 1$. We have $J^{k+1}P_0 \cap \Omega M = J^k(JP_0) \cap \Omega(M/JM) \cap \Omega M = J^k\Omega(M/JM) \cap \Omega M = J^k\Omega M$, where we use that M/JM is Koszul and P_0 is a projective cover of M/JM. $\square$