Auslander representation dimension and radical embeddings

based on the talk by Thorsten Holm (Magdeburg)

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The following definition was introduced by Auslander. Let A be an algebra. By representation dimension of A we mean

$$\mathrm{repdim}(A) := \inf\{\mathrm{gldim}\,\mathrm{End}_A(N) \mid N \text{ is a generator–cogenerator of } \mathrm{mod}\,A\}.$$

It is known that $\mathrm{repdim}(A) = 0$ if and only if A is semisimple. Moreover, for each algebra A, $\mathrm{repdim}(A) \neq 1$, and $\mathrm{repdim}(A) \leq 2$ if and only if A is of finite representation type.

The following new results attracted the interest to representation dimension.

Theorem (Iyama)**.** *Let be an A algebra. If M is an A-module, then there exists an A-module M' such that* $\mathrm{End}_A(M \oplus M')$ *is quasi-hereditary.*

Corollary. *If A is an algebra, then* $\mathrm{repdim}(A) < \infty$*.*

Theorem (Xi)**.** *Let A and B be algebras. If A and B are stable equivalent of Morita type, then* $\mathrm{repdim}(A) = \mathrm{repdim}(B)$*.*

Corollary. *Let A and B be selfinjective algebras. If A and B are derived equivalent, then* $\mathrm{repdim}(A) = \mathrm{repdim}(B)$*.*

Recall that by finitistic global dimension of an algebra A we mean

$$\mathrm{findim}(A) := \sup\{\mathrm{pdim}_A M \mid M \in \mathrm{mod}\,A,\ \mathrm{pdim}_A M < \infty\}.$$

Theorem (Igusa, Todorov)**.** *If* $\mathrm{repdim}(A) \leq 3$*, then* $\mathrm{findim}(A) < \infty$*.*

Let A and B be algebras and let J_A and J_B be the radicals of A and B, respectively. We call a homomorphism $f : A \to B$ a radical embedding if f is a monomorphism and $f(J_A) = J_B$.

Theorem (Erdmann, Holm, Iyama, Schröer). *Let A and B be algebras and let $f : A \to B$ be a radical embedding. If B is of finite representation type, then* $\operatorname{repdim}(A) \leq 3$.

Proof. Let $N_1, \ldots, N_r$ be the complete set of indecomposable B-modules and let $N := A \oplus A^* \oplus f^*(N_1) \oplus \cdots \oplus f^*(N_r)$, where $f^* : \operatorname{mod} B \to \operatorname{mod} A$ is the functor induced by f. In order to show that $\operatorname{gldim} \operatorname{End}_A(N) \leq 3$, we need the following lemma.

Lemma (Auslander). *Let A be an algebra and let N be a generator–cogenerator of* $\operatorname{mod} A$. *Then* $\operatorname{gldim} \operatorname{End}_A(N) \leq 3$ *if and only if for each indecomposable A-module X there exists a short exact sequence*

$$0 \to M_1 \to M_0 \to X \to 0$$

such that $M_0, M_1 \in \operatorname{add}(N)$ and the induced sequence

$$0 \to \operatorname{Hom}_N(N, M_1) \to \operatorname{Hom}_A(N, M_0) \to \operatorname{Hom}_A(N, X) \to 0$$

is exact.

□

Let A be the path algebra of the bound quiver $(Q = (Q_0, Q_1, s, e), I)$ and assume (for simplicity) that I is generated by paths. For a given vertex l of Q we denote by $S(l)$ the set of all arrows in Q which starts in l and by $E(l)$ the set of all arrows in Q which ends in l. Fix a vertex l of Q and assume there are given subsets $S_1, S_2 \subseteq S(l)$ and $E_1, E_2 \subseteq E(l)$ such that $S(l) = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$, $E(l) = E_1 \cup E_2$ and $E_1 \cap E_2 = \varnothing$. We call $Sp = (S_1, S_2, E_1, E_2)$ a splitting datum at l if $\beta\alpha = 0$, for each $\alpha \in E_i$, $\beta \in S_j$, such that $i \neq j$.

Given a splitting datum Sp at a vertex l of Q we define a new quiver $Q^{Sp} = (Q_0^{Sp}, Q_1^{Sp}, s^{Sp}, e^{Sp})$ in the following way. The set of vertices of Q^{Sp} is $\{l_1, l_2\} \cup Q_0 \setminus \{l\}$ (where $l_1, l_2 \notin Q_0$), the set of arrows of Q^{Sp} is just the set of arrows of Q, and given an arrow $\alpha \in Q_1^{Sp}$,

$$s^{Sp}(\alpha) = \begin{cases} s(\alpha) & s(\alpha) \neq l, \\ l_1 & \alpha \in S_1, \\ l_2 & \alpha \in S_2, \end{cases} \quad \text{and } e^{Sp}(\alpha) = \begin{cases} e(\alpha) & e(\alpha) \neq l, \\ l_1 & \alpha \in E_1, \\ l_2 & \alpha \in E_2. \end{cases}$$

Let $I^{Sp} = I$ (note that we may remove from the set of generators of I^{Sp} all paths which contain subpaths of the form $\beta\alpha$, $\alpha \in E_i$, $\beta \in S_j$, $i \neq j$) and $A^{Sp} = kQ^{Sp}/I^{Sp}$.

Proposition. *Let A be the path algebra of a bound quiver (Q, I). If Sp is a splitting datum at a vertex of Q, then there exists a radical embdding $A \hookrightarrow A^{Sp}$.*

An algebra $A = kQ/I$ is called special biserial, if at each vertex of Q starts at most two arrows and ends at most two arrows, and for each arrow β of Q there exists at most one arrow γ such that $\beta\gamma \notin I$ and at most one arrow δ such that $\delta\beta \notin I$. A special biserial algebra $A = kQ/I$ is called a string algebra if I is generated by paths. It is known that given a special biserial algebra, we can obtain a string algebra by factoring out socles of projective-injective modules.

Proposition. *Let A be an algebra and let P be a projective-injective A-module. If* $\operatorname{repdim}(A/\operatorname{soc} P) \leq 3$*, then* $\operatorname{repdim}(A) \leq 3$.

We have the following result.

Theorem. *If A is a special biserial algebra, then* $\operatorname{repdim}(A) \leq 3$.

Proof. Without loss of generality we may assume that $A = kQ/I$ is a string algebra. We show that there exists a racial embedding $A \hookrightarrow B$, where B is of representation finite type. Let

$$c(A) := |\{l \in Q_0 \mid |S(l)| = 2\}| + |\{l \in Q_0 \mid |E(l)| = 2\}|.$$

If $c(A) = 0$, then A is a Nakayama algebra, hence A is representation finite and the claim follows. Now let $c(A) \geq 1$, and let l be such a vertex of Q that $|S(l)| = 2$ (the case of $|E(l)| = 2$ is done analogously). Let $S(l) = \{\alpha_1, \alpha_2\}$. We define the following splitting datum $Sp = (S_1, S_2, E_1, E_2)$ at l: $S_1 = \{\alpha_1\}$, $S_2 = \{\alpha_2\}$, $E_1 = \{\beta \in Q_1 \mid \alpha_2\beta = 0\}$ and $E_2 = E(l) \setminus E_1$. Then A^{Sp} is a string algebra and $c(A^{Sp}) < c(A)$. By induction hypothesis there exists a radical embedding $A^{Sp} \hookrightarrow B$, where B is a representation finite algebra, and since we have a radical embedding $A \hookrightarrow A^{Sp}$, the claim follows. □

Note that using this proof we may explicitly construct a generator–cogenerator N of $\operatorname{mod} A$ such that $\operatorname{gldim} \operatorname{End}_A(N) \leq 3$.

Corollary. *If A is a special biserial algebra, then* $\operatorname{findim}(A) < \infty$.

We have also some other results of the above type.

Proposition. *If S is a Schur algebra of tame representation type, then* $\operatorname{repdim}(S) = 3$.

Proposition. *If A is an algebra of dihedral, semidihedrial or quaterion type, then* $\operatorname{repdim}(A) = 3$.

There is a fundamental open question if there exists an algebra A such that $\operatorname{repdim}(A) \geq 4$?