Derived equivalence of algebras

based on the talk by Torsten Holm (Magdeburg)

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Let k be a fixed algebraically closed field. By an algebra we will mean a finite dimensional algebra over k. The category of A-modules will be denoted by A-mod. The algebras A and B are said to be Morita equivalent provided the categories A-mod and B-mod are equivalent. Morita showed that A and B are Morita equivalent if and only if there exists a progenerator (projective generator) $P \in A$-mod such that $B \simeq \operatorname{End}_A(P)$.

Recall that derived category $D(A)$ of A is the localization of $K(A)$ with respect to quasi-isomorphisms. Note that each A-module is quasi-isomorphic with its projective resolution. We call two algebras A and B derived equivalent if the categories $D(A)$ and $D(B)$ are equivalent. Rickard showed that algebras A and B are derived equivalent if and only if there exists a bounded complex T of projective A-modules such that $\operatorname{Hom}_{K(A)}(T, T[i]) = 0$, $i \neq 0$, $\operatorname{add}(T)$ generate $K(A)$, and $B = \operatorname{End}_{K(A)}(T)$. The complexes of the above type are called tilting complexes. For example, let A be that path algebra of the quiver $1 \leftarrow 2 \leftarrow 3$. Then

$$T = \cdots \to 0 \to P_1 \oplus P_2 \oplus P_2 \xrightarrow{[0,0,i]} P_3 \to 0 \to \cdots$$

where $i : P_2 \to P_3$ is an embedding, is a tilting complex, such that $\operatorname{End}_{K(A)}(T)$ is the path algebra of the quiver $\bullet \leftarrow \bullet \to \bullet$.

It is known that we have the following invariants of derived equivalence:

- the number of simple modules;
- being symmetric or weakly symmetric;
- being of finite global dimension;
- Hochschild (co)homology, cyclic homology;
- center of an algebra.

Moreover, for selfinjective algebras we also know that stable category, representation type and representation dimension are invariant.

It is obvious that any Morita equivalence is a derived equivalence. Happel showd that if T is a tilting A-module, then $\mathrm{End}_A(T)$ and A are derived equivalent.

Asashiba classified all selfinjective algebras of finite representation type with respect to the derived equivalence. Białkowski, Holm and Skowroński classified weakly symmetric algebras of tubular type, and Bocian, Holm and Skowroński classified weakly symmetric algebras of Euclidean type

The following Broue's abelian defect group conjecture seems to be very important. Let G be a finite group and B be a block of kG with abelian defect group D. If b is the corresponding Brauer block in $kN_G(D)$, then B and b should be derived equivalent. There are counterexamples if D is not abelian.

This conjecture is known if D cyclic and if B is a prinipal block with $D \simeq C_3 \times C_3$. The later proof uses the classification of finite simple groups. It is also known for many blocks for simple groups. One can find new results on the following page

`http://www.maths.bris.ac.uk/~majcr/adgc/adgc.html`

Chuang and Rouquier showed that all blocks of symmetric groups with the same weight are derived equivalent (with a block of a symmetric group one can associate its weight). As the corollary it follows that abelian defect group conjecture holds for blocks of symmetric groups.