On simplicial (co)homologies and Hochschild cohomologies of finite dimensional algebras

based on the talk by Stanisław Kasjan

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Throughout the talk k will denote a fixed algebraically closed field. All considered algebras will be basic algebras. We will usually assume that for an algebra A we have fixed a complete set of pairwise orthogonal primitive idempotents e_x, $x \in Q_0$. For $x, y \in Q_0$ by $A(x, y)$ we denote $e_x A e_y$. An algebra A is called schurian if $\dim_k A(x, y) \leq 1$ for all $x, y \in Q_0$.

Let A be schurian. For $\mathbf{x} = (x_0, \dots, x_p) \in Q_0^{p+1}$ by $\phi_{\mathbf{x}}$ we denote the map from $A(x_0, x_1) \times \cdots \times A(x_{p-1}, x_p)$ to $A(x_0, x_p)$ induced by the multiplication in A. Let $S_p(A) = \{\mathbf{x} = (x_0, \dots, x_p) \in Q_0^{p+1} \mid x_0, \dots, x_p \text{ are pairwise different and } \phi_{\mathbf{x}} \neq 0\}$, $p \geq 1$, and let $S_0(A) = Q_0$. Let $C_p(A) = \mathbb{Z}S_p(A)$, $p \geq 0$, and let $d_p : C_p(A) \to C_{p-1}(A)$ be defined by

$$d_p(x_0, \dots, x_p) = \sum_{j=0}^{p} (-1)^j (x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_p).$$

One can show that $d_{p-1} d_p = 0$, thus we get a chain complex $C_*(A)$. By simplicial homology groups of A we mean $SH_p(A) = H_p(C_*(A)) = \operatorname{Ker} d_p / \operatorname{Im} d_{p+1}$.

Let A be schurian. For $x, y \in Q_0$ such that $A(x, y) \neq 0$ we fix $b_{x,y} \in A(x, y)$ with $b_{x,x} = e_x$. Let $B_1 = \{b_{x,y} \mid (x, y) \in S_1(A)\}$ and $C'_p(A)$, $p \geq 1$, be the free abelian group generated by sequences $(b_1, \dots, b_p)$ such that $b_1, \dots, b_p \in B_1$ and $b_1 \cdots b_p \neq 0$. We also put $C'_0(A) = \mathbb{Z}Q_0$. If $b, b' \in B_1$ and $bb' \neq 0$, then $bb' = c_{b,b'} b''$ for uniquely determined $c_{b,b'} \in k^*$ and $b'' \in B_1$. Let $[b, b'] = b''$. We define $d'_p : C'_p(A) \to C'_{p-1}(A)$ be the formula

$$d'_p(b_1, \dots, b_p) = (b_2, \dots, b_p) + \sum_{j=2}^{p} (-1)^{j-1} (b_1, \dots, b_{j-2}, [b_{j-1}, b_j], b_{j+1}, \dots, b_p) + (-1)^p (b_1, \dots, b_{p-1})$$

$p \geq 2$, and $d_1'(b) = y - x$ for $b \in A(x,y)$. It easily follows that the maps $f_p : C_p(A) \to C_p'(A)$ defined by $(x_0, \ldots, x_p) \mapsto (b_{x_0,x_1}, \ldots, b_{x_{p-1},x_p})$, $p \geq 1$, $x \mapsto x$, $p = 0$, form an isomorphism of complexes. Thus $SH_p(A) \simeq H_p(C_*'(A))$.

We still assume that A is schurian. Let $C_p''(A)$, $p \geq 1$, be the free abelian group with basis consisting of the sequences $(b_1, \ldots, b_p)$ such that $b_1, \ldots, b_p \in B$ and $b_1 \cdots b_p \neq 0$, where $B = \{b_{x,y} \mid x, y \in Q_0,\ A(x,y) \neq 0\}$. Let $C_0''(A) = \mathbb{Z}Q_0$. We define d_p'' by the same formulas as d_p' and we get a complex $C_*''(A)$.

Theorem ([1, Section 8.1]). *The inclusion $C_*'(A) \hookrightarrow C_*''(A)$ induces isomorphisms $H_p(C_*'(A)) \simeq H_p(C_*''(A))$.*

Proof. Let $D_p(A)$ be the subgroup of $C_p''(A)$ generated by all $(b_1, \ldots, b_p)$ such that $b_i = e_x$ for some i and x, $p > 0$, $D_0(A) = 0$. Obviously, $C_p''(A) = C_p'(A) \oplus D_p(A)$. Moreover, $d_p''(D_p(A)) \subseteq D_{p-1}(A)$. Thus $C_*''(A) = C_*'(A) \oplus D_*(A)$ and we have to show that $H_p(D_*(A)) = 0$. In order to do it we construct maps $s_p : D_p(A) \to D_{p+1}(A)$ such that $s_{p-1}d_p + d_{p+1}s_p = \mathrm{Id}_{D_p(A)}$, $p > 0$. We put $s_0 = 0$. Let $p \geq 1$ and $\sigma = (b_1, \ldots, b_s, \underbrace{e, \ldots, e}_{k}, d_1, \ldots, d_t) \in D_p(A)$, where $b_1, \ldots, b_s$ are not idempotents, $e = e_x$ for some $x \in Q_0$ and $d_1 \neq e$. Then we put $s_p(\sigma) = (-1)^s \varepsilon(k)(b_1, \ldots, b_s, \underbrace{e, \ldots, e}_{k+1}, d_1, \ldots, d_t)$, where $\varepsilon(k)$ is the remainder from the division of k by 2. □

Let A be an arbitrary algebra. A subset B of A is called a semi-normed basis of A if the following conditions are satisfied:

(1) B is a basis of A,

(2) $B = \bigcup_{x,y \in Q_0} B_{x,y}$, where $B_{x,y} = B \cap A(x,y)$,

(3) $e_x \in B$ for each $x \in Q_0$,

(4) if $b, b' \in B$ then $bb' \in kb''$ for some $b'' \in B$.

Note that if $bb' \neq 0$, then b'' such that $bb' \in kb''$ is uniquely determined by b and b' and we will denote it by $[b, b']$. We will also denote by $c_{b,b'}$ the unique scalar $c \in k^*$ such that $bb' = \lambda[b, b']$. If $bb' = 0$, then we put $[b, b'] = 0$ and $c_{b,b'} = 0$. We can define a chain complex $C_*''(B)$ and simplicial homologies $SH_p(B) = H_p(C_*''(B))$ as above. Obviously, if A is schurian then $SH_p(B) = SH_p(A)$.

Note that $SH_0(B)$ is the free abelian group generated by the blocks of A. In order to calculate $SH_1(B)$, we assume that A is indecomposable. Let $Q_1 = \{b \in B \cap \mathrm{rad}(A) \mid b \neq [b', b''] \text{ for any } b', b'' \in B \cap \mathrm{rad}(A)\}$. It follows that the elements $b + \mathrm{rad}^2(A)$, $b \in Q_1$, form a basis of $\mathrm{rad}(A)/\mathrm{rad}^2(A)$. As the consequence it follows that we may assume that $A = kQ/I$, where the set of vertices of Q is Q_0, the set of arrows of Q is Q_1, $s(b) = x$ and $t(b) = y$ for $b \in Q_1 \cap B_{x,y}$, and I is an admissible ideal in Q. It follows that I is generated by elements of the form u, where u is a path, and $u - \lambda w$, where u, w are paths with the same source and target and $\lambda \in k^*$.

In the above situation the fundamental group $\Pi_1(Q, I)$ of (Q, I) can be defined in the following way: $\Pi_1(Q, I) = \Pi_1(Q)/N(I)$, where $\Pi_1(Q)$ is the fundamental group of Q and $N(I)$ is the normal subgroup of $\Pi_1(Q)$ generated by all the elements of the form $wuv^{-1}w^{-1}$, where w, u, v are paths, $u, v \notin I$ and $u - \lambda v \in I$ for some $\lambda \in k^*$. Let $\omega = \alpha_1^{\varepsilon_1} \cdots \alpha_t^{\varepsilon_t}$ be a loop in Q. We associate with ω an element $\phi_\omega = \sum \varepsilon_i \alpha_i$ of $C_1''(B)$. Since ω is a loop, it follows that $d_1''(\phi_\omega) = 0$. Thus ϕ induces a homomorphism $\Pi_1(Q, I) \to SH_1(B)$, which factorizes through the natural projection $\Pi_1(Q, I) \to \Pi_1(Q, I)/[\Pi_1(Q, I), \Pi_1(Q, I)]$. It follows that the induced map $\Pi_1(Q, I)/[\Pi_1(Q, I), \Pi_1(Q, I)] \to SH_1(B)$ is an isomorphism.

Note that if A is a triangular and representation finite algebra, then A is schurian.

Theorem ([2]). *Let A be a triangular and representation finite algebra. Then $SH_1(A)$ is a free abelian group and $SH_p(A) = 0$, $p > 1$.*

Let B be a semi-normed basis of an algebra A and let Z be an abelian group. We define the p-th cohomology group $SH^p(B, Z)$ in coefficients in Z as $H^p(\mathrm{Hom}_{\mathbb{Z}}(C_*''(B), Z))$. The universal coefficients theorem states, that there exists an exact sequence $0 \to \mathrm{Ext}^1_{\mathbb{Z}}(SH_{p-1}(B), Z) \to SH^p(B, Z) \to \mathrm{Hom}_{\mathbb{Z}}(SH_p(B), Z) \to 0$ of groups, which splits. In particular, $SH^1(B, Z) = \mathrm{Hom}_{\mathbb{Z}}(SH_1(B), Z)$.

Note that associativity of the multiplication in A implies that the map $(b, b') \mapsto c_{b,b'}$ induces a 2-cocycle in $\mathrm{Hom}_{\mathbb{Z}}(C_*''(B), k^*)$. On the other hand, if $d_{b,b'}$, $b, b' \in B$, $bb' \neq 0$, are elements of k^* such that the function induced by the assignment $(b, b') \mapsto d_{b,b'}$ is a 2-cocycle, then the multiplication defined in A by the formula $(b, b') \mapsto d_{b,b'}[b, b']$, where $d_{b,b'} = 0$ if $bb' = 0$, is associative. If in addition, $d_{e_x,b} = c_{e_x,b}$ and $d_{b,e_x} = c_{b,e_x}$ for all b and x, then in this way we obtain an algebra A' with the same set of primitive orthogonal idempotents as A and the semi-normed basis B. Bretscher and Gabriel proved in [2] that there exists an isomorphism $\lambda : A \to A'$ such that $\lambda b \in kb$ for $b \in B$ if and only if the above 2-cocycles induce the same element in $SH^2(B, k^*)$.

We say that a semi-normed basis B of A is multiplicative if $bb' = [b, b']$ for all $b, b' \in B$.

Corollary. *If $SH^2(B, k^*) = 0$, then A has a multiplicative basis.*

Proof. The function $(b,b') \mapsto 1$ if $bb' \neq 0$ and $(b,b') \mapsto 0$ otherwise, induces a 2-cocycle in $\mathrm{Hom}_{\mathbb{Z}}(C''_*(B), k^*)$, thus we obtain an algebra A' with the multiplicative basis B. Since $SH^2(B,k^*) = 0$ is follows that A and A' are isomorphic, thus A also admits a multiplicative basis. $\square$

Corollary. *If A is a triangular and representation finite algebra, then A has a multiplicative basis.*

Proof. We have to show that $SH^2(A,k^*) = 0$. According to the universal coefficients theorem we have an exact sequence $0 \to \mathrm{Ext}^1_{\mathbb{Z}}(SH_1(A), k^*) \to SH^2(A,k^*) \to \mathrm{Hom}_{\mathbb{Z}}(SH_2(A), k^*) \to 0$. Since $SH_1(A)$ is a free abelian group, thus $\mathrm{Ext}^1_{\mathbb{Z}}(SH_1(A), k^*) = 0$. Moreover, we have $SH_2(A) = 0$, hence $\mathrm{Hom}_{\mathbb{Z}}(SH_2(A), k^*) = 0$, and consequently $SH^2(A,k^*) = 0$. $\square$

One can show (see [1]) that every representation finite algebra has a multiplicative basis.

Exercise. Show directly, that if A is a triangular and representation finite algebra, then $SH_2(A)$ is a torsion group.

For an algebra A and an A-A-bimodule M, we can consider a complex

$$0 \to \mathrm{Hom}_k(k, M) \xrightarrow{\partial^0} \mathrm{Hom}_k(A, M) \xrightarrow{\partial^1} \mathrm{Hom}_k(A \otimes_k A, M) \xrightarrow{\partial^2} \cdots,$$

where $(\partial^p(f))(a_1, \ldots, a_{p+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{p+1}) + \sum_{j=1}^{p} (-1)^j f(a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_{p+1}) + (-1)^{p+1} f(a_1 \otimes \cdots \otimes a_p) a_{p+1}$ (in particular, $\delta^0(f)(a) = af(1) - f(1)a$). We call $HH^p(A, M) = \mathrm{Ker}\, \delta^p / \mathrm{Im}\, \delta^{p-1}$ the pth Hochschild cohomology group of A with coefficients in M. We denote $HH^p(A, A)$ by $HH^p(A)$.

Let B be a semi-normed basis of an algebra A. For $p \geq 1$ we define a map $\varepsilon_p : \mathrm{Hom}_{\mathbb{Z}}(C''_p(B), k) \to \mathrm{Hom}_k(A^{\otimes^p}, A)$ defined by the formula $\varepsilon_p(f)(b_1 \otimes \cdots \otimes b_p) = f(b_1, \ldots, b_p) b_1 \ldots b_p$ if $(b_1, \ldots, b_p) \in C''_p(B)$ and $\varepsilon_p(f)(b_1 \otimes \cdots \otimes b_p) = 0$ otherwise. We also put $(\varepsilon_0(f))(1) = \sum_{x \in Q_0} f(x) e_x$ for $f \in \mathrm{Hom}_{\mathbb{Z}}(C''_p(B), k)$. One checks that ε_p, $p \geq 0$, induce a homomorphism of complexes, thus we have the induced k-linear homomorphisms $\xi_p : SH^p(B, k) \to HH^p(A)$.

It follows that if A is schurian, then ξ_p, $p \geq 0$, are monomorphisms. Indeed, ξ_0 is a monomorphism since ε_0 is a monomorphism. Let $p \geq 2$ and take $h \in \mathrm{Ker}\, d^*_{p+1}$. Let $f = \xi_p(h)$ and assume $f = \partial^{p-1}(g)$ for some $g \in \mathrm{Hom}_k(A^{\otimes^{p-1}}, A)$. Let $(b_1, \ldots, b_{p-1}) \in C''_{p-1}(B)$ with $b_1 \in A(x, s)$ and $b_{p-1} \in A(t, y)$.

Then $e_x g(b_1 \otimes \cdots \otimes b_{p-1}) e_y = \mu(b_1, \ldots, b_{p-1}) b_1 \cdots b_{p-1}$ for an elements $\mu(b_1, \ldots, b_{p-1})$ of k, since A is schurian. Denote by μ the induced map $\mathbb{Z}C''_{p-1} \to k$. It follows that $d^*_p(\mu) = h$.

We have the following theorem.

Theorem. *Let $\tilde{A} \to A$ be a Galois covering with a free group G such that $\tilde{A}$ is schurian. Then* $\dim_k HH^1(A) \geq \operatorname{rk} G$.

Proof. Let $\tilde{A} = k\tilde{Q}/\tilde{I}$ and $A = kQ/I$ and assume that $(\tilde{Q}, \tilde{I})$ is a Galois covering of (Q, I). Then G is a quotient of $\Pi_1(Q, I)$. Since $\tilde{A}$ is schurian, it follows that $\tilde{A}$ has a semi-normed basis. Because G is free, one can choose a G-invariant semi-normed basis. It induces a semi-normed basis B of A such that $\alpha + I \in B$ for every arrow $\alpha \in Q_1$. Then $\Pi_1(Q, I)/[\Pi_1, \Pi_1] \simeq SH_1(B)$. As the consequence, $G/[G, G]$ is a quotient of $SH_1(B)$, hence $\operatorname{rk} G \leq \operatorname{rk} SH_1(B)$. Then $\operatorname{rk} G \leq \operatorname{rk} SH_1(B) \leq \dim_k \operatorname{Hom}_{\mathbb{Z}}(SH_1(B), k) = \dim_k SH^1(B, k) \leq \dim_k HH^1(A)$. $\square$

Let A be schurian and triangular. For $s \in Q_0$, let $\overline{A}^s$ be the incidence algebra of the poset $\{t \in Q_0 \mid A(s,t) \neq 0\}$, where $t \leq t'$ if and only if $A(s,t)A(t,t') \neq 0$. Let $A^{(s)} = A \setminus \{s\}$ and $\overline{A}^{(s)} = \overline{A}^s \setminus \{s\}$. Dually we define $\underline{A}_t$, $A_{(t)}$ and $\underline{A}_{(t)}$. Note that $SH_n(\overline{A}^s) = SH_n(\underline{A}_t) = 0$ for all $n > 0$. We will say than an algebra A has no suspended crown if $SH_1(\underline{D}_{(t)}) = 0$ for every $s, t \in Q_0$ such that $A(s,t) \neq 0$ and for every full subcategory D of $\overline{A}^s$. It follows that if $\operatorname{gldim} A \leq 2$, then A has no suspended crown.

Let A be a schurian and triangular algebra and let s be a source in A. If $(x_0, \ldots, x_p) \in C_p(A)$, then either $x_0 = s$ and $(x_0, \ldots, x_p) \in C_p(\overline{A}^s)$ or $x_0 \neq s$ and $(x_0, \ldots, x_p) \in C_p(A^{(s)})$. Thus we have an epimorphism $C_p(\overline{A}^s) \oplus C_p(A^{(s)}) \to C_p(A)$ with the kernel $C_p(\overline{A}^{(s)})$. In this way we get an exact sequence of complexes $0 \to C_*(\overline{A}^{(s)}) \to C_*(\overline{A}^s) \oplus C_*(A^{(s)}) \to C_*(A) \to 0$, which induces the following long exact sequence of homologies

$$\begin{aligned} \cdots \to SH_p(\overline{A}^{(s)}) \to SH_p(\overline{A}^s) \oplus SH_p(A^{(s)}) \to SH_p(A) \\ \to SH_{p-1}(\overline{A}^{(s)}) \to SH_{p-1}(\overline{A}^s) \oplus SH_{p-1}(A^{(s)}) \to \cdots, \end{aligned}$$

which we call the Mayer–Vietoris sequence.

Assume in addition that A has no suspended crown. We want to show that $SH_2(A)$ free. First note that if s is a source then $SH_2(\overline{A}^{(s)}) = 0$. Indeed, we prove by induction that $SH_2(D) = 0$ for all full subcategories D of $\overline{A}^{(s)}$. If t is a target in D, then using the dual Mayer–Vietoris sequence we get an exact sequence

$$SH_2(\underline{D}_t) \oplus SH_2(D_{(t)}) \to SH_2(D) \to SH_1(\underline{D}_{(t)}).$$

Note that $SH_2(\underline{D}_t) = 0$ by general observations, $SH_2(D_{(t)}) = 0$ by the induction hypothesis, and $SH_1(\underline{D}_{(t)}) = 0$ by assumptions on A, hence the claim follows.

Now we show that if s is a source, then $SH_1(\overline{A}^{(s)})$ is a free abelian group. We again use an induction to prove that $SH_1(D)$ is a free abelian group for each full subcategory D of $\overline{A}^{(s)}$. Using the Mayer–Vietoris sequence we have

$$SH_1(\underline{D}_{(t)}) \to SH_1(\underline{D}_t) \oplus SH_1(D_{(t)}) \to SH_1(D) \to SH_0(\underline{D}_{(t)}).$$

Here we have $SH_1(\underline{D}_{(t)}) = 0$, $SH_1(\underline{D}_t) = 0$, $SH_1(D_{(t)})$ is a free abelian group and $SH_0(\underline{D}_{(t)})$ is a free abelian group, which implies the claim.

Now we can show that $SH_2(A)$ is free. We again show it for each full subcategory D of A. We have the following exact sequence

$$SH_2(\overline{D}^{(s)}) \to SH_2(\overline{D}^s) \oplus SH_2(D^{(s)}) \to SH_2(D) \to SH_1(\overline{D}^{(s)}).$$

We know that $SH_2(\overline{D}^{(s)}) = 0$, $SH_2(\overline{D}^s) = 0$, $SH_2(D^{(s)})$ is a free abelian grup and $SH_1(D^{(s)})$ is a free abelian group, hence the claim follows.

Theorem. *Let A be a schurian and triangular algebra such that* $\operatorname{gldim} A \leq 2$ *and $HH^2(A) = 0$. Then A has a multiplicative basis.*

Proof. Since $HH^2(A) = 0$, hence $\mathrm{Hom}_{\mathbb{Z}}(SH_2(A), k) = 0$, and consequently $SH_2(A) = 0$, because $SH_2(A)$ is a free abelian group. Thus $SH^2(A, k^*) = \mathrm{Ext}^1_{\mathbb{Z}}(SH_1, k^*) = 0$, as k^* is divisible. Now the claim follows. $\square$

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