REPRESENTATION DIMENSION AND RADICALS

BASED ON THE TALK BY TIMO ROSNAU

The talk was based on the paper *On the representation dimension of finite dimensional algebras* by Changchang Xi.

Definition.

Let V and M be modules over a finite dimensional algebra A. A homomorphism $f : X \to M$ is called a right $\operatorname{add} V$-approximation of M if $X \in \operatorname{add} Y$ and $\operatorname{Hom}_A(Y, f)$ is surjective for each $Y \in \operatorname{add} V$.

Lemma.

If V and M are modules over a finite dimensional algebra A, then there exists a right $\operatorname{add} V$-approximation of M. Moreover, if $A \in \operatorname{add} V$, then each right $\operatorname{add} V$-approximation is surjective.

Lemma (Auslander).

Let V be a generator-cogenerator of $\operatorname{mod} A$ for a finite dimension algebra A and $m \in \mathbb{N}_+$. Then $\operatorname{gl.dim} \operatorname{End}_A(V)^{\mathrm{op}} \leq m + 2$ if and only if for each indecomposable A-module M there exists an exact sequence

$$0 \to X_0 \to \cdots \to X_m \to M \to 0$$

such that $X_0, \ldots, X_m \in \operatorname{add} V$ and the sequence

$$0 \to \operatorname{Hom}_A(X, X_0) \to \cdots \to \operatorname{Hom}_A(X, X_m) \to \operatorname{Hom}_A(X, M) \to 0$$

is exact for each $X \in \operatorname{add} V$.

Theorem.

Let A be a finite dimensional algebra, $n \in \mathbb{N}$ be such that $\mathfrak{r}_\Lambda^n = 0$, and $B := A/\mathfrak{r}_\Lambda^{n-1}$. If $I/\mathfrak{r}_\Lambda^{n-1} I \in \operatorname{add}(B \oplus DB)$ for each injective A-module I, then

$$\operatorname{rep.dim} A \leq \max(3, \operatorname{rep.dim} B + 1).$$

Assumptions.

Let $m = \operatorname{rep.dim} B$. It is known that $m < \infty$. If $m \leq 2$, then the result is known, thus we may assume $m \geq 3$. Choose a generator-cogenerator U of $\operatorname{mod} B$ such that $\operatorname{gl.dim} \operatorname{End}_B(U)^{\mathrm{op}} = m$. Let $V := A \oplus DA \oplus U$.

Lemma.

Let $I \in \operatorname{add} DA$ and M be an indecomposable A-module. If either $M \in \operatorname{mod} B$ or $M \notin \operatorname{add} DA$, then $\operatorname{Hom}_A(p_I, M)$ is an isomorphism, where $p_I : I \to I/\mathfrak{r}_\Lambda^{n-1} I$ is the canonical projection.

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Proof.

If $\mathfrak{r}_\Lambda^{n-1} I = 0$, then the claim is obvious, thus assume $\mathfrak{r}_\Lambda^{n-1} I \neq 0$. Then $\mathfrak{r}_\Lambda^{n-1} I = \operatorname{soc} I$ is simple. Moreover, our assumptions imply there are no monomorphism $I \to M$, hence $\mathfrak{r}_\Lambda^{n-1} I$ is contained in the kernel of every homomorphism $I \to M$, hence the claim follows.

Lemma.

If M is an indecomposable B-module, then there exists there exists an exact sequence

$$0 \to X_0 \to \cdots \to X_{m-2} \to M \to 0$$

such that $X_0, \ldots, X_{m-2} \in \operatorname{add} U$ and the sequence

$$0 \to \operatorname{Hom}_A(X, X_0) \to \cdots \to \operatorname{Hom}_A(X, X_{m-2}) \to \operatorname{Hom}_A(X, M) \to 0$$

is exact for each $X \in \operatorname{add} V$.

Proof.

By Auslander's Lemma there exists a sequence

$$0 \to X_0 \to \cdots \to X_{m-2} \to M \to 0$$

such that $X_0, \ldots, X_{m-2} \in \operatorname{add} U$ and the sequence

$$0 \to \operatorname{Hom}_A(X, X_0) \to \cdots \to \operatorname{Hom}_A(X, X_{m-2}) \to \operatorname{Hom}_A(X, M) \to 0$$

is exact for each $X \in \operatorname{add} U$. It remains to show that the above sequence is exact for each $X \in \operatorname{add} V$. We may obviously assume that X is indecomposable. If either $X \in \operatorname{add} U$ or $X \in \operatorname{add} A$, then the claim is clear. Thus assume that $X \in \operatorname{add} DA$. The previous lemma implies that it is enough to show that the sequence

$$0 \to \operatorname{Hom}_A(X/\mathfrak{r}_\Lambda^n X, X_0) \to \cdots \to \operatorname{Hom}_A(X/\mathfrak{r}_\Lambda^n X, X_{m-2}) \\ \to \operatorname{Hom}_A(X/\mathfrak{r}_\Lambda^n X, M) \to 0$$

is exact. However, this follows since $X/\mathfrak{r}_\Lambda^n X \in \operatorname{add}(B \oplus DB) \subset \operatorname{add} U$.

Proof (of Theorem).

We show that $\operatorname{gl.dim} \operatorname{End}_A(V)^{\mathrm{op}} \leq m+1$ using Auslander's Lemma, i.e. for each indecomposable A-module M we construct an exact sequence

$$0 \to X_0 \to \cdots \to X_{m-1} \to M \to 0$$

such that $X_0, \ldots, X_{m-1} \in \operatorname{add} V$ and the sequence

$$0 \to \operatorname{Hom}_A(X, X_0) \to \cdots \to \operatorname{Hom}_A(X, X_{m-1}) \to \operatorname{Hom}_A(X, M) \to 0$$

is exact for each $X \in \operatorname{add} V$. Obviosuly, we may assume that $M \notin \operatorname{add} V$. Let $M' := \{m \in M \mid \mathfrak{r}_\Lambda^{n-1} m = 0\}$. Observe that $M' \in \operatorname{mod} B$. Let $l : X_m \to M'$ be a right $\operatorname{add} U$-approximation of M', $g : P \to M/M'$ be the A-projective cover of M/M', $h : P \to M$ be a lift of g, and $f := [il, h] : X_m \oplus P \to M$, where $i : M' \to M$ is the canonical injection. One easily checks that f is surjective and $\operatorname{Ker} f \in \operatorname{mod} B$. Thus it follows from the above lemma that in order to finish the proof it suffices to show that $\mathrm{Hom}_A(X, f)$ is surjective for each $X \in \operatorname{add} V$. This is clear for $X \in \operatorname{add} A$. On the other hand, if $X \in \operatorname{add} U$, then $\mathrm{Hom}_A(X, il)$ is surjective, since $\mathrm{Hom}_A(X, i)$ is an isomorphism, thus the claim also follows in this case. It remains to consider the case $X \in \operatorname{add} DA$. Again, it suffices to show that $\mathrm{Hom}_A(X, il)$ is surjective. Since $X_m \in \operatorname{mod} B$ and M is indecomposable and not in $\operatorname{add} DA$, this is equivalent to surjectivity of $\mathrm{Hom}_A(X/\mathfrak{r}_\Lambda^{n-1}X, il)$, which follows, since $X/\mathfrak{r}_\Lambda^{n-1}X \in \operatorname{add}(B \oplus DB) \subset \operatorname{add} U$.