REPRESENTATION DIMENSION AND GLOBAL DIMENSION

BASED ON THE TALK BY NILS MAHRT

The talk was based on the paper *On the representation dimension of finite dimensional algebras* by Changchang Xi.

THEOREM.
Let A be a finite dimensional algebra. If $\mathrm{Fac}(DA)$ is of finite representation type and $\mathrm{Hom}_A(X,Y) = 0$ for all indecomposable A-modules X and Y such that $X \in \mathrm{Fac}(DA)$ and $Y \notin \mathrm{Fac}(DA)$, then $\mathrm{rep.dim}\, A \leq \mathrm{gl.dim}\, A + 2$.

PROOF.
Choose an A-module N such that $\mathrm{add}\, N = \mathrm{Fac}(DA)$. Let $V := A \oplus N$. We may assume that $\mathrm{gl.dim}\, A < \infty$. According to Auslander's lemma it is enough to show that for each indecomposable A-module M there exists an exact sequence

$$0 \to X_0 \to \cdots \to X_n \to M \to 0$$

such that $X_0, \dots, X_n \in \mathrm{add}\, V$ and the sequence

$$0 \to \mathrm{Hom}_A(X, X_0) \to \cdots \to \mathrm{Hom}_A(X, X_n) \to \mathrm{Hom}_A(X, M) \to 0$$

is exact for each $X \in \mathrm{add}\, V$, where $n := \mathrm{pd}_A\, M$. If $M \in \mathrm{add}\, V$, then the claim is obvious, thus we may assume that $M \notin \mathrm{add}\, V$. Let $f : P \to M$ be the projective cover of M. Since $\mathrm{pd}_A \mathrm{Ker}\, f = n - 1$ (or $\mathrm{Ker}\, f = 0$ if $n = 0$), it remains to prove that $\mathrm{Hom}_A(X, f)$ is surjective for each indecomposable $X \in \mathrm{add}\, V$. This is obvious is $X \in \mathrm{add}\, DA$. Moreover, if $X \in \mathrm{Fac}\, DA$, then $\mathrm{Hom}_A(X, M) = 0$ and the claim follows.

COROLLARY.
If A is an hereditary finite dimensional algebra, then $\mathrm{rep.dim}\, A \leq 3$.

COROLLARY.
If A is a tame concealed algebra, then $\mathrm{rep.dim}\, A \in \{3, 4\}$.

REMARK (RINGEL).
One may show that if $\mathrm{Fac}(DA)$ is of finite representation type for a finite dimensional algebra A, then $\mathrm{rep.dim}\, A \leq 3$. This implies that if A is tame concealed, then $\mathrm{rep.dim}\, A = 3$.

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Lemma.

Let A be a finite dimensional algebra. If N is an A-A-bimodule and $B := \left[\begin{smallmatrix} A & N \\ 0 & A \end{smallmatrix}\right]$, then $\operatorname{gl.dim} B \leq \operatorname{gl.dim} A + \operatorname{pd}_A N + 1$.

Proof.

Let $I := \left[\begin{smallmatrix} 0 & N \\ 0 & A \end{smallmatrix}\right]$ and $J := \left[\begin{smallmatrix} A & N \\ 0 & 0 \end{smallmatrix}\right]$. Observe that I and J are ideals in B and $IJ = 0$. In particular, if $M \in \operatorname{mod} B$, then JM is an B/I-module. Since $B/I \simeq A$ and B/I is a projective B-module, it follows that $\operatorname{pd}_B(JM) \leq \operatorname{gl.dim} A$. On the other hand, M/JM is a B/J-module. Again $B/J \simeq A$. Moreover, we have a short exact sequence $0 \to N \to I \to B/J$, hence $\operatorname{pd}_B(B/J) \leq \operatorname{pd}_B N + 1$, since I is a projective B-module. Using that $\operatorname{pd}_B N \leq \operatorname{pd}_A N$, we obtain our claim.

Proposition.

If A is a finite dimensional algebra and $\operatorname{Hom}_A(DA, A) = 0$, then $\operatorname{rep.dim} A \leq 1 + 2 \operatorname{gl.dim} A$.

Proof.

We take $V := A \oplus DA$ and use the above lemma.