RADICAL EMBEDDINGS AND REPRESENTATION DIMENSION

BASED ON THE TALK BY CLAUS MICHAEL RINGEL

The talk was inspired by the paper *Radical embeddings and representation dimension* by Karin Erdmann, Thorsten Holm, Osamu Iyama and Jan Schröer.

Definition.
A bound quiver (Q, I) is called special biserial if the following conditions are satisfied:

(1) for each $x \in Q_0$ the number of $\alpha \in Q_1$ such that $s\alpha = x$ ($t\alpha = x$) is at most 2,
(2) for each $\alpha \in Q_1$ the number of $\beta \in Q_1$ such that $\alpha\beta \notin I$ ($\beta\alpha \notin I$) is at most 1.

Definition.
A special biserial quiver (Q, I) is called string if I is generated by paths.

Definition.
A module M over an algebra A is called local if $\operatorname{top} M$ is simple.

Lemma.
If M is a representation of a string bound quiver (Q, I), then M is a direct sum of local string modules if and only if $\alpha M \cap \beta M = 0$ for all $\alpha, \beta \in Q_1$, $\alpha \neq \beta$.

Definition.
A module M over an algebra A is called torsionless if M is a submodule of a projective A-module.

Definition.
An algebra A is called torsionless finite if there is only a finite number of the isomorphism classes of indecomposable torsionless A-modules.

Theorem.
Any string algebra is torsionless finite.

Proof.
We first observe that if M is a torsionless representation of a string bound quiver (Q, I), then $\alpha M \cap \beta M = 0$ for all $\alpha, \beta \in Q_1$, $\alpha \neq \beta$. Next we use the above lemma and the fact that there is only a finite number of local string modules.

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DEFINITION.

Let (Q, I) be a string bound quiver. For $x \in Q_0$ such that the number of $\alpha \in Q_1$ with $s\alpha = x$ is 2 we define the bound quiver (Q', I') called the s-splitting of (Q, I) at x as follows. Let α_1 and α_2 be the arrows in Q with $s\alpha = x$. We put $Q'_0 := Q_0 \setminus \{x\} \cup \{x_0, x_1, x_2\}$ and $Q'_1 := Q_1$. Moreover,

$$s'\alpha := \begin{cases} s\alpha & s\alpha \neq x, \\ x_1 & \alpha = \alpha_1, \\ x_2 & \alpha = \alpha_2 \end{cases} \quad \text{and} \quad t'\alpha := \begin{cases} t\alpha & t\alpha \neq x, \\ x_0 & t\alpha = x \text{ and } \alpha_1\alpha, \alpha_2\alpha \in I, \\ x_1 & t\alpha = x \text{ and } \alpha_1\alpha \notin I, \\ x_2 & t\alpha = x \text{ and } \alpha_2\alpha \notin I. \end{cases}$$

Finally, $I' := I$.

Dually, we define the t-splitting of a string bound quiver (Q, I) at $x \in Q_0$ such that the number of $\alpha \in Q_1$ with $t\alpha = x$ is 2.

LEMMA.

Let (Q, I) be a string algebra and $x \in Q_0$ be such that the number of $\alpha \in Q_1$ with $s\alpha = x$ is 2. If (Q', I') is the s-splitting of (Q, I) at x, then $kQ/I \to kQ'/I'$ given by

$$y \mapsto \begin{cases} y & y \neq x, \\ x_1 + x_2 + x_3 & y = x, \end{cases} x \in Q_0, \qquad \text{and} \qquad \alpha \mapsto \alpha, \ \alpha \in Q_1,$$

is a well-defined radical embedding.

THEOREM.

If A is a string algebra, then there exists a radical embedding $A \to B$ with B a Nakayama algebra.

COROLLARY.

If A is a string algebra, then $\operatorname{rep.dim} A \leq 3$.

REMARK.

The above corollary follows from any of the above two theorems.

COROLLARY.

If A is a special biserial algebra, then $\operatorname{rep.dim} A \leq 3$.

PROOF.

We prove it by induction on $\dim_k A$. If A is a string algebra, then the claim follows from the previous corollary. Otherwise, there exists a projective-injective A-module P. Since A/P is special biserial, $\operatorname{rep.dim}(A/P) \leq 3$ by induction, and this implies $\operatorname{rep.dim} A \leq 3$.