A GENERALIZATION OF TITLING MODULES

BASED ON THE TALK BY LINGLING YAO

ASSUMPTION.
Throughout the talk R will be a fixed ring.

DEFINITION.
Let U be an R-module and $n \in \mathbb{N}$. An R-module M is called n-presented by U if there exists an exact sequence of the form

$$U^{(X_{n-1})} \to \cdots \to U^{(X_0)} \to M \to 0.$$

NOTATION.
For an R-module U and $n \in \mathbb{N}$ we denote by $\mathrm{Pres}_n(U)$ the full subcategory of the category of R-modules of modules which n-presented by U.

REMARK.
If U is an R-module, then $\mathrm{Pres}_0(U)$ coincides with the category of R-modules.

REMARK.
If U is an R-module, then $\mathrm{Pres}_1(U)$ is the category $\mathrm{Gen}(U)$ of modules generated by U.

REMARK.
If U is an R-module, then $\mathrm{Pres}_2(U)$ is the category $\mathrm{Pres}(U)$ of modules presented by U.

DEFINITION.
An R-module U is called self-small if the canonical map

$$\mathrm{Hom}_R(U, U)^{(X)} \to \mathrm{Hom}_R(U, U^{(X)})$$

is an isomorphism for each set X.

DEFINITION.
Let $n \in \mathbb{N}_+$. An R-module U is called n-quasi-projective if for any exact sequence of the form

$$0 \to M \to U^{(X)} \to N \to 0$$

with $M \in \mathrm{Pres}_{n-1}(U)$, the induced sequence

$$0 \to \mathrm{Hom}_R(U, M) \to \mathrm{Hom}_R(U, U^{(X)}) \to \mathrm{Hom}_R(U, N) \to 0$$

is exact.

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Notation.

Let U be an R-module and $S := \mathrm{End}_R(U)$. For each R-module M we define

$$\rho_M^U : U \otimes_S \mathrm{Hom}_R(U, M) \to M$$

by $\rho_M^U(u \otimes f) := f(u)$ for $u \in U$ and $f \in \mathrm{Hom}_R(U, M)$.

Definition.

Let U be an R-module. An R-module M is called U-static if ρ_M^U is an isomorphism.

Notation.

For an R-module U we denote by $\mathrm{Stat}(U)$ the full subcategory of the category of R-modules of U-static modules.

Notation.

Let U be an R-module and $S := \mathrm{End}_R(U)$. For each R-module M we define

$$\delta_M^U : M \to \mathrm{Hom}_S(\mathrm{Hom}_R(M, U), U)$$

by $(\delta_M(m))(f) := f(m)$ for $m \in M$ and $f \in \mathrm{Hom}_R(M, U)$.

Definition.

Let U be an R-module. An R-module M is called U-reflexive if δ_M^U is an isomorphism.

Notation.

For an R-module U we denote by $\mathrm{Refl}(U)$ the subcategory of the category of R-modules of U-reflexive modules.

Definition.

Let U be an R-module. An R-module M is called U-torsionless if δ_M^U is a monomorphism.

Remark.

Let U be an R-module. An R-module M is U-torsionless if and only if M belongs to the subcategory $\mathrm{Cogen}(U)$ of the category of R-modules of modules cogenerated by U.

Definition.

Let V be an R-module, $S := \mathrm{End}_R(V)$, I be an injective cogenerator of the category of R-modules, and $W := \mathrm{Hom}_R(V, C)$. We say that V is a $*$-module if $\mathrm{Hom}_R(V, -)$ and $V \otimes_S -$ induce quasi-inverse equivalences between the categories $\mathrm{Gen}(V)$ and $\mathrm{Cogen}(W)$.

Remark.

An R-module V is tilting if and only if V is a $*$-module and the injective envelope of R belongs to $\mathrm{Gen}(V)$.

Theorem.

The following conditions are equivalent for an R-module V.

(1) V is a $*$-module.
(2) V is self-small and 2-quasi-projective, and $\mathrm{Pres}(V) = \mathrm{Gen}(V)$.
(3) V is self-small, and for any $W \in \mathrm{Add}\, V$ and any submodule M of W the following holds: $M \in \mathrm{Gen}(V)$ if and only if the canonical map $\mathrm{Ext}^1_R(V, M) \to \mathrm{Ext}^1_R(V, W)$ is injective.
(4) V is self-small and for any exact sequence

$$0 \to L \to M \to N \to 0$$

with $M, N \in \mathrm{Gen}(V)$ the following holds: $L \in \mathrm{Gen}(V)$ if and only if the sequence

$$0 \to \mathrm{Hom}_R(V, L) \to \mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(V, N) \to 0$$

is exact.

Definition.

Let $n \in \mathbb{N}_+$. An R-module V is called an n-$*$-module if V is self-small and $(n+1)$-quasi-projective, and $\mathrm{Pres}_{n+1}(V) = \mathrm{Pres}_n(V)$.

Definition.

An R-module V is called a static-$*$-module if V is self-small and for any exact sequence

$$0 \to L \to M \to N \to 0$$

with $M, N \in \mathrm{Stat}(V)$ the following holds: $L \in \mathrm{Stat}(V)$ if and only if the sequence

$$0 \to \mathrm{Hom}_R(V, L) \to \mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(V, N) \to 0$$

is exact.