SKEW GROUP CATEGORIES AND DERIVED EQUIVALENCES

BASED ON THE TALK BY HIDETO ASASHIBA

Definition.

Let $\mathscr{C}$ be a category with an action of a group G. If $F : \mathscr{C} \to \mathscr{C}'$ is a functor, then a family $\phi = (\phi_\alpha : F \to F\alpha)_{\alpha \in G}$ of natural isomorphisms is called admissible if the following conditions are satisfied:

(1) $\phi_{1,x} = \mathrm{Id}_{Fx}$ for each $x \in \mathscr{C}$,
(2) $\phi_{\beta\alpha,x} = \phi_{\beta,\alpha x}\phi_{\alpha,x}$ for each $\alpha, \beta \in G$ and $x \in \mathscr{C}$.

Notation.

Let $\mathscr{C}$ be a category with an action of a group G, $F : \mathscr{C} \to \mathscr{C}'$ a functor, and $\phi = (\phi_\alpha : F \to F\alpha)_{\alpha \in G}$ an admissible family of natural isomorphisms. For all $x, y \in \mathscr{C}$ we define

$$F^{(1)}_{x,y} : \bigoplus_{\alpha \in G} \mathscr{C}(\alpha x, y) \to \mathscr{C}'(Fx, Fy)$$

and

$$F^{(2)}_{x,y} : \bigoplus_{\alpha \in G} \mathscr{C}(x, \alpha y) \to \mathscr{C}'(Fx, Fy)$$

by

$$F^{(1)}_{x,y}((f_\alpha)_{\alpha \in G}) := \sum_{\alpha \in G} F f_\alpha \circ \phi_{\alpha,x} \text{ and } F^{(2)}_{x,y}((f_\alpha)_{\alpha \in G}) := \sum_{\alpha \in G} \phi_{\alpha^{-1},\alpha y} \circ F f_\alpha.$$

Proposition.

Let $\mathscr{C}$ be a category with an action of a group G, $F : \mathscr{C} \to \mathscr{C}'$ a functor, and $\phi = (\phi_\alpha : F \to F\alpha)_{\alpha \in G}$ an admissible family of natural isomorphisms. If $x, y \in \mathscr{C}$, then $F^{(1)}_{x,y}$ is isomorphism if and only if $F^{(2)}_{x,y}$ is an isomorphism.

Definition.

Let $\mathscr{C}$ be a category with an action of a group G, $F : \mathscr{C} \to \mathscr{C}'$ a functor, and $\phi = (\phi_\alpha : F \to F\alpha)_{\alpha \in G}$ an admissible family of natural transformations. The pair (F, ϕ) is called G-precovering if $F^{(2)}_{x,y}$ is an isomorphism for each $x, y \in \mathscr{C}$. If in addition, F is dense then we say that the pair (F, ϕ) is covering.

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Definition.

Let $\mathscr{C}$ be a category with an action of a group G. We define the category $\mathscr{C}/G$ as follows: $\mathscr{C}/G$ has the same objects as $\mathscr{C}$ and

$$\begin{aligned}(\mathscr{C}/G)(x,y) := \{f \in \prod_{\alpha,\beta\in G} \mathscr{C}(\alpha x, \beta y) \mid\ & f_{\gamma\alpha,\gamma\beta} = \gamma f_{\alpha,\beta} \text{ for } \alpha,\beta,\gamma \in G,\\ & \#\{\beta \in G \mid f_{\alpha,\beta} \neq 0\} < \infty \text{ for each } \alpha \in G, \text{ and}\\ & \#\{\alpha \in G \mid f_{\alpha,\beta} \neq 0\} < \infty \text{ for each } \beta \in G\}.\end{aligned}$$

We have the canonical functor $\pi : \mathscr{C} \to \mathscr{C}/G$ defined by

$$\pi(x) := x \qquad \text{and} \qquad \pi(f) := (\delta_{\alpha,\beta}\alpha f)_{\alpha,\beta}$$

and the admissible family $\phi = (\phi_\mu : \pi \to \pi\mu)_{\mu\in G}$ of natural isomorphisms defined by

$$\phi_{\mu,x} = (\delta_{\alpha,\beta\mu} \operatorname{Id}_{\alpha x})_{\alpha,\beta}.$$

The pair (π, ϕ) is a G-covering functor.

Remark.

If $\mathscr{C}$ is a category with an action of a group G, then

$$(\mathscr{C}/G)(x,y) \simeq \bigoplus_{\alpha\in G} \mathscr{C}(x, \alpha y).$$

Definition.

A category $\mathscr{C}$ is called basis if $x \not\simeq y$ for all $x, y \in \mathscr{C}$, $x \neq y$.

Definition.

A category $\mathscr{C}$ is called semi-perfect if $\mathscr{C}(x,x)$ is local for each $x \in \mathscr{C}$.

Lemma.

If $\mathscr{C}$ is a locally finite dimensional basic and semi-perfect category with an action of a group G, then

$$\dim_k (\mathscr{C}/G)(x,x)/\operatorname{rad}(\mathscr{C}/G)(x,x) = |G_x|$$

for each $x \in \mathscr{C}$.