IYAMA'S FINITENESS THEOREM (VIA STRONGLY QUASI-HEREDITARY ALGEBRAS)

BASED ON THE TALK BY CLAUS MICHAEL RINGEL

Notation.

For an artin algebra Γ we denote by $\mathscr{S}(\Gamma)$ the collection of the simple Γ-modules.

Definition.

If Γ is an artin algebra, then we call a function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$ admissible provided $l(S') = l(S'')$ for all $S', S'' \in \mathscr{S}(\Gamma)$ with $S' \simeq S''$.

Notation.

If Γ is an artin algebra with an admissible function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$, then we put $n_\Gamma := |l(\mathscr{S}(\Gamma))|$.

Notation.

Let Γ be an artin algebra with an admissible function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$. For $S \in \mathscr{S}(\Gamma)$ we define $P(S)$, $\Delta(S)$, and $R(S)$ in the following way: $P(S)$ is the projective cover of S, $\Delta(S)$ is the maximal quotient of $P(S)$ such that $l(T) \leq l(S)$ for each simple composition factor T of $\Delta(S)$, and $R(S)$ is the kernel of the canonical projection map $P(S) \to \Delta(S)$.

Remark.

Let Γ be an artin algebra with an admissible function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$. If $S \in \mathscr{S}(\Gamma)$ and there exists an exact sequence $0 \to R \to P(S) \to \Delta \to 0$ such that $l(T) > l(S)$ for each direct summand T of $\operatorname{top} R$ and $l(T) \leq l(S)$ for each simple composition factor T of Δ, then $R \simeq R(S)$ and $\Delta \simeq \Delta(S)$.

Definition.

Let Γ be an artin algebra together with an admissible function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$. We call Γ strongly quasi-hereditary (with respect to l) if for each $S \in \mathscr{S}(\Gamma)$ the following conditions are satisfied: $l(T) < l(S)$ for each simple composition factor T of $\operatorname{rad} \Delta(S)$ and $R(S)$ is a projective Γ-module.

Remark.

If Γ is an artin algebra with an admissible function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$, then Γ is strongly quasi-hereditary if and only if for each $S \in \mathscr{S}(\Gamma)$ there exists an exact sequence $0 \to R \to P(S) \to \Delta \to 0$ such that the following conditions are satisfied: $R \simeq P(T_1) \oplus \cdots \oplus P(T_m)$ for some $T_1, \ldots, T_m \in$

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$\mathscr{S}(\Gamma)$ with $l(T_1), \dots, l(T_m) > l(S)$, and $\operatorname{Hom}_\Lambda(P(T), \operatorname{rad}\Delta) = 0$ for each $T \in \mathscr{S}(\Gamma)$ with $l(T) \geq l(S)$.

Lemma.

If an artin algebra Γ with an admissible function $l : \mathscr{S}(\Gamma) \to \mathbb{N}$ is strongly quasi-hereditary, then $\operatorname{gl.dim}\Gamma \leq n_\Gamma$.

Notation.

For a module X over an artin algebra Λ we put

$$\partial X := \{f(x) \mid f \in \operatorname{rad}_\Lambda(X, X),\ x \in X\}.$$

More generally, we put $\partial^0 X := X$ and $\partial^i X := \partial\partial^{i-1} X$ for $i \in \mathbb{N}_+$. Moreover, $n(X) := \min\{i \in \mathbb{N} \mid \partial^i X = 0\}$. Finally, $\mathscr{C}_i(X) := \operatorname{add}(\partial^i X \oplus \cdots \oplus \partial^{n(X)-1} X)$ for $i \in \mathbb{N}$.

Lemma.

Let X be a module over an artin algebra Λ and let N be an indecomposable Λ-module such that $N \in \operatorname{add}\partial^{i-1} X \setminus \operatorname{add}\partial^i X$ for some $i \in \mathbb{N}_+$. If

$$\alpha(N) := \{f(x) \mid f \in \operatorname{rad}_\Lambda(\partial^{i-1} X, N),\ x \in \partial^{i-1} X\},$$

then the natural inclusion map $\alpha(N) \to N$ is a right $\mathscr{C}_i(X)$-approximation.

Proof.

Obviously, $\alpha(N) \in \mathscr{C}_i(X)$, thus it remains to show that $\operatorname{Im} f \subset \alpha(N)$ for each $f \in \operatorname{Hom}_\Lambda(\partial^j X, N)$ with $j \in [i, \infty)$. There exist epimorphisms $g : (\partial^i X)^p \to \partial^j X$ and $h : (\partial^{i-1} X)^q \to (\partial^i X)^p$ for some $p, q \in \mathbb{N}$. Since $N \notin \operatorname{add}\partial^i X$, $fgh \in \operatorname{rad}_\Lambda((\partial^{i-1} X)^q, N)$, and consequently $\operatorname{Im}(fgh) \subset \alpha(N)$. Since additionally gh is an epimorphism, the claim follows.

Theorem.

Let X be a module over an artin algebra Λ. If M is a Λ-module such that $\mathscr{C}_0(X) = \operatorname{add} M$, then Γ is strongly quasi-hereditary with $n_\Gamma = n(X)$, where $\Gamma := \operatorname{End}_\Lambda(M)^{\mathrm{op}}$.

Proof.

Put $\mathscr{C}_i := \mathscr{C}_i(X)$ for $i \in \mathbb{N}$. Moreover, for a Λ-module L let $H_L := \operatorname{Hom}_\Lambda(M, L)$. Recall that the projective Γ-modules are of the form H_L for $L \in \operatorname{add} M$. Moreover, $\operatorname{Hom}_\Gamma(H_{L'}, H_{L''}) \simeq \operatorname{Hom}_\Lambda(L', L'')$ for all $L', L'' \in \operatorname{add} M$. In particular, $\operatorname{Hom}_\Gamma(H_{L'}, \operatorname{rad} H_{L''}) \simeq \operatorname{rad}_\Lambda(L', L'')$ for all $L', L'' \in \operatorname{add} M$.

Fix indecomposable $N \in \operatorname{add} M$ and let $i \in \mathbb{N}_+$ be such that $N \in \mathscr{C}_{i-1} \setminus \mathscr{C}_i$. Put

$$\alpha(N) := \{f(x) \mid f \in \operatorname{rad}_\Lambda(\partial^{i-1} X, N),\ x \in \partial^{i-1} X\}.$$

Let $\mathscr{C}_i(M, N)$ denotes the space of $f \in \operatorname{Hom}_\Lambda(M, N)$ which factors through an object in $\mathscr{C}_i$. Observe that the previous lemma implies that the sequence $0 \to H_{\alpha(N)} \to H_N \to H_N/\mathscr{C}_i(M,N) \to 0$ is exact. Moreover $\alpha(N) \in \mathscr{C}_i$. Thus in order to finished the proof, it remains to show that $\operatorname{Hom}_\Gamma(H_L, \operatorname{rad} H_N/\mathscr{C}_i(M,N)) = 0$ for each indecomposable Λ-module $L \in \mathscr{C}_{i-1}$. Fix such L. The claim is obvious if $L \in \mathscr{C}_i$, hence we may assume that $L \in \operatorname{add} \partial^{i-1} X$. If $\xi \in \operatorname{Hom}_\Gamma(H_L, \operatorname{rad} H_N/\mathscr{C}_i(M,N))$, then there exists $f \in \operatorname{rad}_\Lambda(L,N)$ such that $\xi(g) = fg + \mathscr{C}_i(M,N)$ for each $g \in H_L$. Since $L \in \operatorname{add} \partial^{i-1} X$, $\operatorname{Im} f \subset \alpha(N)$, and consequently $\xi = 0$.

Corollary.

If X is a module over an artin algebra Λ, then there exists a Λ-module M such that $\operatorname{gl.dim} \operatorname{End}_\Lambda(M) \leq n(X)$.