COMPLEXITY AND REPRESENTATION DIMENSION

BASED ON THE TALK BY ROLF FARNSTEINER

Assumption.

Throughout the talk Λ is the group algebra of a finite group G over a field k.

Notation.

Let $A := \bigoplus_{n\in\mathbb{N}} H^{2n}(G,k)$. Recall that $H^n(G,k) := \underline{\mathrm{Hom}}_\Lambda(\Omega^n(k),k)$ for each $n \in \mathbb{N}$. For $M \in \operatorname{mod}\Lambda$ we define $\Phi_M : A \to \operatorname{Ext}^*_\Lambda(M,M)$ by $\Phi_M(f) := \underline{f \otimes \mathrm{Id}_M}$. Recall that $\operatorname{Ext}^n_\Lambda(M,N) \simeq \underline{\mathrm{Hom}}_\Lambda(\Omega^n(M),N)$ for all $M,N \in \operatorname{mod}\Lambda$ and $n \in \mathbb{N}_+$.

Remark.

Observe that A is a notherian k-algebra. Furthermore, Friedlander and Suslin proved that if $M, N \in \operatorname{mod}\Lambda$, then the actions of A on $\operatorname{Ext}^*_\Lambda(M,N)$ induced by Φ_M and Φ_N coincide, and $\operatorname{Ext}^*_\Lambda(M,N)$ is a finitely generated A-module.

Definition.

If $M \in \operatorname{mod}\Lambda$, then we define the support variety $\mathscr{V}(M)$ of M as the zero set $Z(\operatorname{Ker}\Phi_M)$ of $\operatorname{Ker}\Phi_M$. The complexity $\operatorname{cx}(M)$ of M equals by definition $\dim\mathscr{V}(M)$.

Definition.

If $\zeta \in H^n(G,k)$ for $n \in \mathbb{N}$, then the Carlson module L_ζ equals by definition $\operatorname{Ker}\hat{\zeta}$ for a representative $\hat{\zeta}$ of ζ in $\mathrm{Hom}_\Lambda(\Omega^n(k),k)$.

Theorem.

If $\zeta \in H^{2n}(G,k)$ for $n \in \mathbb{N}$ and $M \in \operatorname{mod}\Lambda$, then $\mathscr{V}(M \otimes_k L_\zeta) = \mathscr{V}(M) \cap Z(\zeta)$, where $Z(\zeta)$ is the zero set of ζ.

Notation.

If $\mathscr{C}$ and $\mathscr{D}$ are full subcategories of $\underline{\operatorname{mod}}\Lambda$, then $\mathscr{C} * \mathscr{D}$ denotes the additive subcategory of $\underline{\operatorname{mod}}\Lambda$ generated by $M \in \underline{\operatorname{mod}}\Lambda$ such that there exists a distinguished triangle $C \to M \to D \to C[1]$ with $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

Lemma (Rouquier).

Let $H_1, \ldots, H_r : \underline{\operatorname{mod}}\Lambda \to Ab$ be cohomological functors. For each $i \in [1, r-1]$, let $f_i : H_i \to H_{i+1}$ be a natural transformation. If $\mathscr{C}_1$, $\ldots$, $\mathscr{C}_{r-1}$ are full subcategories of $\underline{\operatorname{mod}}\Lambda$ closed under shifts such that

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for all $i \in [1, r-1]$ and $C \in \mathscr{C}_i$ there exists $n \in \mathbb{N}$ such that $(f_i)_{C[j]} = 0$ for each $j \geq n$, then for each $X \in \mathscr{C}_{r-1} * \cdots * \mathscr{C}_1$ there exists m such that $(f_{r-1} \cdots f_1)_{X[j]} = 0$ for each $j \gg m$.

Notation.

If $M \in \operatorname{mod} \Lambda$ and $d \in \mathbb{N}_+$, then

$$\langle M \rangle_d := \underbrace{\operatorname{add} M * \cdots * \operatorname{add} M}_{d \text{ times}}.$$

Lemma.

Let $M \in \operatorname{mod} \Lambda$. If there exists a chain of morphisms

$$M = K_r \xrightarrow{g_{r-1}} K_{r-1} \to \cdots \to K_2 \xrightarrow{g_1} K_1$$

such that $\underline{g_{r-1} \cdots g_1} \neq 0$, $\operatorname{cx}(K_1) = 1$, and $\operatorname{Ext}_\Lambda^j(g_i, M) = 0$ for all $i \in [1, r-1]$ and $j \gg 0$, then $K_1 \notin \langle M \rangle_{r-1}$.

Proof.

Let $H_i := \underline{\operatorname{Hom}}_\Lambda(K_i, -)$ for $i \in [1, r]$ and $f_i := \underline{\operatorname{Hom}}(g_i, -)$ for $i \in [1, r-1]$. The last condition implies that $(f_i)_{M[j]} = 0$ for all $i \in [1, r-1]$ and $j \gg 0$. Since $\operatorname{cx}(K_1) = 1$, there exists $n \in \mathbb{N}_+$ such that $K_1 \simeq \Omega^{-n}(K_1)$. Then $(f_{r-1} \cdots f_1)_{X[nj]} \neq 0$ for all $j \in \mathbb{N}_+$, thus $K_1 \notin \langle M \rangle_{r-1}$ according to the previous lemma.

Proposition.

If $M \in \operatorname{mod} \Lambda$, then $\langle M \rangle_{\operatorname{cx}(M)-1} \neq \underline{\operatorname{mod}} \Lambda$.

Proof.

We define $K_1, \ldots, K_r$ satisfying the assumptions of the above lemma inductively as follows. Obviously $K_r := M$. If $i \in [2, r]$ and K_i is defined, then we choose homogeneous $\zeta \in A$ such that ξ is not a zero-divisor of $\operatorname{Ext}_\Lambda^*(M \oplus K_i, M \oplus K_i)$ and $\Phi_{M \oplus K_i}(\zeta) : \operatorname{Ext}_\Lambda^n(M \oplus K_i, M \oplus K_i) \to \operatorname{Ext}_\Lambda^{n+|\zeta|}(M \oplus K_i, M \oplus K_i)$ is injective for all $n \gg 0$. Then we put $K_{i-1} := \Omega^{-1}(K_i \otimes L_\zeta)$ and one shows that we can apply the previous lemma.

Definition.

We put

$$\dim \underline{\operatorname{mod}} \Lambda := \min\{d \in \mathbb{N} \mid \langle M \rangle_{d+1} = \underline{\operatorname{mod}} \Lambda \text{ for some } M \in \operatorname{mod} \Lambda\}.$$

Theorem.

$\dim \underline{\operatorname{mod}} \Lambda \geq \operatorname{cx}(\Lambda / \operatorname{rad} \Lambda) - 1$.

Proof.

Let $d = \dim \underline{\operatorname{mod}} \Lambda$. If $M \in \operatorname{mod} \Lambda$ is such that $\langle M \rangle_{d+1} = \underline{\operatorname{mod}} \Lambda$, then $d \geq \operatorname{cx}(M) - 1$ according to the previous lemma. On the other hand, $\operatorname{cx}(M) \geq \operatorname{cx}(\Lambda / \operatorname{rad} \Lambda)$, and the claim follows.

Corollary.

$\operatorname{rep.dim} \Lambda \geq \operatorname{cx}(\Lambda / \operatorname{rad} \Lambda) + 1$.

Remark.

The above theorems can be extended to selfinjective algebras with finite cohomology. A selfinjective algebra Λ has finite cohomology if there exists a noetherian algebra A together with a ring homomorphism $\Phi_M : A \to \operatorname{Ext}^*_\Lambda(M, M)$ for each Λ-module M, such that for all $M, N \in \operatorname{mod}\Lambda$ the A-module structures on $\operatorname{Ext}^*_\Lambda(M, N)$ induced by Φ_M and Φ_N coincide and $\operatorname{Ext}^*_\Lambda(M, N)$ is a finitely generated A-module.