COMPLEXITY AND THE DIMENSION OF A TRIANGULATED CATEGORY, I

BASED ON THE TALK BY ANGELA HOLTMANN

The talk was based on the paper *Complexity and the dimension of a triangulated category* by Petter Andreas Bergh and Steffen Oppermann.

Assumption.

Throughout the talk k denotes a fixed artin commutative ring.

Definition.

For a triangulated category $\mathscr{T}$ we define the dimension $\dim \mathscr{T}$ of $\mathscr{T}$ by

$$\dim \mathscr{T} := \inf\{d \in \mathbb{N} \mid \text{there exists } M \in \mathscr{T} \text{ such that } \mathscr{T} = \langle M \rangle_{d+1}\}.$$

Definition.

A positively graded k-module V is said to be of finite type if $\ell(V_n) < \infty$ for all $n \in \mathbb{N}$.

Definition.

For a positively graded k-module V of finite type we define the rate of growth $\gamma(V)$ by

$$\gamma(V) := \inf\{t \in \mathbb{N} \mid \text{there exists } a \in \mathbb{R} \text{ such that } \ell(V_n) \leq a n^{t-1} \text{ for all } n \gg 0.\}$$

Notation.

For objects X and Y of a triangulated category $\mathscr{T}$ we put

$$\operatorname{Hom}^+_{\mathscr{T}}(X, Y) := \bigoplus_{n \in \mathbb{N}} \operatorname{Hom}_{\mathscr{T}}(X, Y[n]).$$

Definition.

For objects X and Y of a triangulated category $\mathscr{T}$ we define the complexity $\operatorname{cx}_{\mathscr{T}}(X, Y)$ by

$$\operatorname{cx}_{\mathscr{T}}(X, Y) := \gamma(\operatorname{Hom}^+_{\mathscr{T}}(X, Y)).$$

Lemma.

If X and Y are objects of a triangulated category $\mathscr{T}$, then

$$\operatorname{cx}_{\mathscr{T}}(X, Y) = \operatorname{cx}_{\mathscr{T}}(X[i], Y[j])$$

for all $i, j \in \mathbb{Z}$.

Date: 18.04.2008.

LEMMA.

If $X_1 \to X_2 \to X_3 \to X_1[1]$ is a distinguished triangle in a triangulated category $\mathscr{T}$, then

$$\mathrm{cx}_{\mathscr{T}}(X_2, Y) \leq \max(\mathrm{cx}_{\mathscr{T}}(X_1, Y), \mathrm{cx}_{\mathscr{T}}(X_3, Y))$$

and

$$\mathrm{cx}_{\mathscr{T}}(Y, X_2) \leq \max(\mathrm{cx}_{\mathscr{T}}(Y, X_1), \mathrm{cx}_{\mathscr{T}}(Y, X_3))$$

for each $Y \in \mathscr{T}$.

LEMMA.

If X_1, X_2 and Y are objects of a triangulated category $\mathscr{T}$, then

$$\mathrm{cx}_{\mathscr{T}}(X_1 \oplus X_2, Y) = \max(\mathrm{cx}_{\mathscr{T}}(X_1, Y), \mathrm{cx}_{\mathscr{T}}(X_2, Y))$$

and

$$\mathrm{cx}_{\mathscr{T}}(Y, X_1 \oplus X_2) = \max(\mathrm{cx}_{\mathscr{T}}(Y, X_1), \mathrm{cx}_{\mathscr{T}}(Y, X_2)).$$

LEMMA.

Let X and Y be objects of a triangulated category $\mathscr{T}$. If $Z \in \langle X \rangle_n$ for some $n \in \mathbb{N}_+$, then

$$\mathrm{cx}_{\mathscr{T}}(Z, Y) \leq \mathrm{cx}_{\mathscr{T}}(X, Y) \qquad \text{and} \qquad \mathrm{cx}_{\mathscr{T}}(Y, Z) \leq \mathrm{cx}_{\mathscr{T}}(Y, X).$$

DEFINITION.

For a finitely generated module M over an artin k-algebra Λ we define the complexity $\mathrm{cx}_\Lambda(M)$ of M by

$$\mathrm{cx}_\Lambda(M) := \inf\{t \in \mathbb{N}_0 \mid \text{there exists } a \in \mathbb{R} \text{ such that } \ell(P_n) \leq an^{t-1} \text{ for all } n \gg 0\},$$

where

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

is the minimal projective resolution of M.

REMARK.

If M is a finitely generated module over an artin k-algebra Λ, then

$$\mathrm{cx}_\Lambda(M) = \mathrm{cx}_{\mathscr{D}^b(\Lambda)}(M, \Lambda/\operatorname{rad}\Lambda).$$

DEFINITION.

We say that a k-artin algebra Λ satisfies the condition (Fg) if there exists a commutative noetherian graded k-algebra H of finite type such that the following holds:

(1) for each $M \in \operatorname{mod}\Lambda$ there exists a graded ring homomorphism $\varphi_M : H \to \operatorname{Ext}^*_\Lambda(M, M)$,
(2) if $M, N \in \operatorname{mod}\Lambda$, then the actions of H on $\operatorname{Ext}^*_\Lambda(M, N)$ via φ_M and φ_N coincide and $\operatorname{Ext}^*_\Lambda(M, N)$ is a finitely generated H-module with respect to this action.

Definition.

The center $Z_{\mathscr{T}}$ of a triangulated category $\mathscr{T}$ is a $\mathbb{Z}$-graded commutative ring such that for each $n \in \mathbb{Z}$ $Z_{\mathscr{T}}[n]$ consists of the natural transformations $f : \mathrm{Id} \to \mathrm{Id}[n]$ such that $f_{X[1]} = f_X[1]$ for each $X \in \mathscr{T}$.

Remark.

If X and Y are objects of a triangulated category $\mathscr{T}$, then $Z_{\mathscr{T}}$ acts on $\mathrm{Hom}^+_{\mathscr{T}}(X, Y)$ by $f * g := f_Y[m]g$ for $f \in Z_{\mathscr{T}}[n]$, $n \in \mathbb{Z}$, and $g \in \mathrm{Hom}_{\mathscr{T}}(X, Y[m])$, $m \in \mathbb{N}$.

Definition.

We say that a triangulated category $\mathscr{T}$ satisfies the condition (Fgc) is there exists a commutative noetherian graded k-algebra H together with a graded ring homorphism $H \to Z_{\mathscr{T}}$ such that $\mathrm{Hom}^+_{\mathscr{T}}(X, Y)$ is a finitely generated H-module for all $X, Y \in \mathscr{T}$.

Proposition.

Let $\mathscr{T}$ be a triangulated category satisfying the condition (Fgc). If M and C are objects of $\mathscr{F}$ such that $c := \mathrm{cx}_{\mathscr{T}}(M, C) > 1$, then there exists a sequence

$$M = K_c \xrightarrow{f_{c-1}} K_{c-1} \to \cdots \to K_2 \xrightarrow{f_1} K_1$$

such that the following conditions are satisfied:

(1) $\mathrm{cx}_{\mathscr{T}}(K_j, C) = j$ for each $j \in [1, r-1]$,
(2) $f_1 \cdots f_{c-1} \neq 0$,
(3) $\mathrm{Hom}_{\mathscr{T}}(f_j, M[i]) = 0$ for each $j \in [1, r-1]$ and $i \gg 0$.

Definition.

An object X of a triangulated category $\mathscr{T}$ is called periodic if there exists $n \in \mathbb{N}_+$ such that $X[n] \simeq X$.

Definition.

An object C of a triangulated category $\mathscr{T}$ is called a periodicity generator if $\mathrm{cx}_{\mathscr{T}}(X, C) = 1$ implies that X is periodic for each $X \in \mathscr{T}$.

Theorem.

If $\mathscr{T}$ is a triangulated category satisfying the condition (Fgc), then

$$\dim \mathscr{T} \geq \sup\{\mathrm{cx}_{\mathscr{T}}(X, C) \mid X \in \mathscr{T}\} - 1$$

for each periodicity generator $C \in \mathscr{T}$.

Proof.

Let $d := \dim \mathscr{T}$. Obviously, we may assume that $d < \infty$. Fix $M \in \mathscr{T}$ such that $\mathscr{T} = \langle M \rangle_{d+1}$. Let $C \in \mathscr{T}$ be a periodicity generator and $c := \mathrm{cx}_{\mathscr{T}}(M, C)$. If $c \leq 1$, then there is nothing to prove, thus assume that $c > 1$. Then there exists a map $f : M \to K$ such that $\mathrm{cx}_{\mathscr{T}}(K, C) = 1$, $f \neq 0$, and for each $X \in \langle M \rangle_{c-1}$ there exists $m \in \mathbb{N}$

with $\mathrm{Hom}_{\mathscr{T}}(f, X[i]) = 0$ for each $i \geq m$. Since C is a periodicity generator, it follows that K is periodic, hence there exists an isomorphism $g : K \to K[n]$ for some $n \in \mathbb{N}_+$. Consequently $\mathrm{Hom}_{\mathscr{T}}(f, K[ni]) \neq 0$ for all $i \in \mathbb{N}_+$, thus $K \notin \langle M \rangle_{c-1}$, what finishes the proof.