ASYMMETRY OVER SELFINJECTIVE ALGEBRAS

BASED ON THE TALK BY DAVID JORGENSEN

Assumption.

Throughout the talk R is a local noetherian commutative ring with the maximal ideal $\mathfrak{m}$ and the residue field k.

Remark.

Let R be Gorenstein. If R is an Artin algebra, then R is selfinjective.

Example.

Let R be Gorenstein and let $\mathbf{x} = (x_1, \dots, x_e)$ be a minimal generating set of $\mathfrak{m}$. If $H_i(\mathbf{x})$ denotes the i-th Koszul homology group, then $H_i(\mathbf{x}) \simeq H_{e-i}(\mathbf{x})$ for each $i \in [0, e]$.

Example.

Let $R = Q/I$ for an ideal I in a regular local ring Q and let $\mathbb{F}$ be the minimal free resolution of R over Q. If R is Gorenstein, then $\mathbb{F}^* \simeq \mathbb{F}$, where $(-)^* := \operatorname{Hom}_Q(-, Q)$.

Example.

Assume that R is 0-dimensional and graded. If R is Gorenstein, then the Hilbert series H^R of R is symmetric.

Example.

Let $R = k[[T^S]]$ for a semigroup $S \subseteq \mathbb{N}$. Then R is Gorenstein if and only if S is symmetric.

Example.

Let R be Gorenstein. If M is a finitely generated R-module, then $\operatorname{pd}_R M < \infty$ if and only if $\operatorname{id}_R M < \infty$. In other words, $\operatorname{Ext}^i_R(M, k) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(k, M) = 0$ for all $i \gg 0$.

Example.

If R is Gorenstein, then R is a dualizing module.

Example.

If R is Gorenstein, then every maximal Cohen–Macaulay module M has a complete resolution, i.e. there exists a minimal ($\operatorname{Im} \partial_i \subseteq \mathfrak{m}F_i$ for each $i \in \mathbb{Z}$) acyclic complex $\mathbb{F}$ of free modules such that $M \simeq \operatorname{Coker} \partial_1$ and $\mathbb{F}^*$ is acyclic. Let b_i denote the rank of F_i for $i \in \mathbb{Z}$. Then (b_i) and (b_{-i}) grow on the same scale if and only if $(\dim_k \operatorname{Ext}^i_R(M, k))$ and $(\dim_k \operatorname{Ext}^i_R(M^*, k))$ grow by the same scale.

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Theorem (Avramov/Buchweitz).

If M and N are finitely generated modules over a complete intersection R, then the following conditions are equivalent:

(1) $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$.
(2) $\operatorname{Ext}^i_R(N,M) = 0$ for all $i \gg 0$.
(3) $\operatorname{Tor}^R_i(M,N) = 0$ for all $i \gg 0$.
(4) $V(M) \cap V(N) = 0$.

Remark.

Let M be a finitely generated module over a complete intersection R. Then $\operatorname{Ext}^*_R(M,k)$ has a structure of a finitely generated module over a polynomial ring $\mathscr{R}$. The cone defined by the annihilator of $\operatorname{Ext}^*_R(M,k)$ is called the support variety of M and denoted $V(M)$. The dimension of $V(M)$ is denoted $\operatorname{cx}_R(M)$ and called the complexity of M. It is known that $(\dim_k \operatorname{Ext}^i_R(M,k))$ grows polynomially of degree $\operatorname{cx}_R(M) - 1$. Avramov and Buchweitz proved that $V(M) = V(M^*)$, hence $(\dim_k \operatorname{Ext}^i_R(M,k))$ and $(\dim_k \operatorname{Ext}^i_R(M^*,k))$ grow by the same scale. One can also prove it using the notion of reducible complexity developed by Bergh.

Theorem (Jorgensen/Şega).

There exists a selfinjective algebra R such that $R = k[x_1, \ldots, x_6]/I$ for a homogeneous ideal I, $H^R = 1 + 6t + 6t^2 + t^3$, and which admit finitely generated modules M and N with the following properties:

(1) $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, but $\operatorname{Ext}^i_R(N,M) \neq 0$ for all $i > 0$.
(2) $(\dim_k \operatorname{Ext}^i_R(M,k)) = (2)$ while $(\dim_k \operatorname{Ext}^i_R(M^*,k))$ has an exponent grow.