ACCESSIBLE ALGEBRAS AND SPECTRAL ANALYSIS

BASED ON THE TALK BY JOSE ANTONIO DE LA PEÑA

Assumption.
Throughout the talk k is a fixed algebraically closed field.

Definition.
A module M over an algebra A is called exceptional if $\operatorname{End}_k(M) = k$ and $\operatorname{Ext}^i_A(M, M) = 0$ for all $i \in \mathbb{N}_+$.

Definition.
We call an algebra A accessible if there exists a sequence of algebras $k = A_0, \ldots, A_s = A$ such that for each $i \in [1, s]$, $A_i = A_{i-1}[M_{i-1}]$ for an exceptional A_{i-1}-module M_{i-1}.

Example.
If A is a tree algebra, then A is accessible.

Fact.
If A is an accessible algebra, then A is triangular. In particular, $\operatorname{gl.dim} A < \infty$.

Fact.
If A is an accessible algebra, then $\operatorname{HH}^0(A) = k$ and $\operatorname{HH}^i(A) = 0$ for all $i \in \mathbb{N}_+$. It follows immediately by considering Happel long exact sequence of Hochschild cohomolgies.

Fact.
If $d \in \mathbb{N}_+$, then there are only finitely many accessible algebras of dimension d.

Fact.
If A is an accessible algebra, then A is a smooth point of the scheme of algebras, since $\operatorname{HH}^3(A) = 0$.

Notation.
For an algebra A of finite global dimension we denote by C_A the Cartan matrix of A.

Definition.
For an algebra A of finite global dimension we put $\varphi_A := -C_A^{\mathrm{tr}} C_A$ and call it the Coxeter transformation of A.

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Notation.

For an algebra A of finite global dimension we put $\chi_A := \det(t \cdot \mathrm{Id} - \varphi_A)$.

Notation.

For an algebra A of finite global dimension we denote by $\operatorname{Spec} \varphi_A$ the spectrum of φ_A.

Definition.

For an algebra A of finite global dimension we put $\rho_A := \max\{|\lambda| \mid \lambda \in \operatorname{Spec} \varphi_A\}$ and call it the spectral radius of A.

Fact.

If A is an algebra of finite global dimension, then $[t]\chi_A = 1$.

Proof.

Note that $[t]\chi_A = -\operatorname{Tr} \varphi_A$. Moreover, according to Happel

$$\operatorname{Tr} \varphi_A = -\sum_{i \in \mathbb{N}} (-1)^i \dim_k \mathrm{HH}^i(A),$$

hence the claim follows.

Fact.

If A is an algebra of finite global dimension, then $\chi_A(-1) = m^2$ for some $m \in \mathbb{N}$. Moreover, if A has an odd number of vertices, then $\chi_A(-1) = 0$.

Proof.

Easy calculations show $\chi_A(-1) = \det(C_A^{\mathrm{tr}} - C_A)$. Since $C_A^{\mathrm{tr}} - C_A$ is skew-symmetric, the claim follows by using the normal forms of skew-symmetric matrices.

Example.

Let A be a hereditary algebra. Then A is tame if and only if $\rho_A = 1$. Moreover, A is of finite representation type if and only if $\rho_A = 1 \notin \operatorname{Spec} \varphi_A$.

Proposition.

For an accessible algebra A the following conditions are equivalent.

(1) A is derived of Dynkin type.
(2) The Euler quadratic form of A is positive definite.
(3) There exists a sequence of algebras $k = A_0, \ldots, A_s = A$ such that for each $i \in [1, s]$, $A_i = A_{i-1}[M_{i-1}]$ for an exceptional A_{i-1}-module M_{i-1}, and $\rho_A = 1 \notin \operatorname{Spec} \varphi_A$.

Proof.

We only prove (3) $\Rightarrow$ (1). Assume that A is not derived of Dynkin type and let $k = A_0, \ldots, A_s = A$ be a sequence of algebras such that for each $i \in [1, s]$, $A_i = A_{i-1}[M_{i-1}]$ for an exceptional A_{i-1}-module

M_{i-1}. Fix $i \in [1, s]$ such that $A_0, \ldots, A_{i-1}$ are derived of Dynkin type and A_i is not derived of Dynkin type. Then it follows that A_i is derived equivalent to $B[P]$ for a projective module P over a hereditary algebra B of Dynkin type. Consequently, A_i is derived equivalent to a representation infinite hereditary algebra and the claim follows.

PROPOSITION.

Let A be a canonical algebra of type $(m_1, \ldots, m_n)$.

(1) A is accessible if and only if $n = 3$.

(2) $\chi_A(-1) = 4$ if $2 \nmid m_i$ for each $i \in [1, m]$, and $\chi_A(-1) = 0$, otherwise.

PROOF.

Since $\dim_k \mathrm{HH}^2(A) = n - 3$, the first part follows. For the second part it is enough to use that

$$\chi_A = (t-1)^2 \prod_{i \in [1,n]} (1 + t + \cdots + t^{m_i - 1}).$$

PROPOSITION.

If A is a representation finite accessible algebra, then Γ_A is a preprojective quiver of tree type.

THEOREM.

For a representation finite algebra A the following conditions are equivalent.

(1) A is accessible.

(2) A is strongly simply connected.

(3) Γ_A is a preprojective quiver of tree type and $\mathrm{HH}^1(A) = 0$.

EXAMPLE.

Let A_n be the path algebra of the quiver

$$\underset{1}{\bullet} \xrightarrow{x} \underset{2}{\bullet} \xrightarrow{x} \cdots \xrightarrow{x} \underset{n-1}{\bullet} \xrightarrow{x} \underset{n}{\bullet}$$

bound by $x^3 = 0$. Then A_{11} is derived equivalent to the canonical algebra of type $(2, 3, 7)$. This implies that $1 = \rho_{A_{12}} \notin \operatorname{Spec} \varphi_{A_{12}}$.

THEOREM.

If A is an admissible algebra which is derived representation finite, then

$$\chi_A(-1) = \begin{cases} 1 & A \text{ has an even number of simple modules,} \\ 0 & A \text{ has an odd number of simple modules.} \end{cases}$$

PROOF.

We assume that A is representation finite. Let $A = B[M]$ with M exceptional. If $M^{\perp}$ is equivalent to $\operatorname{mod} C$ for an algebra C, then $\chi_A = (1 + t) \cdot \chi_B + t \cdot \chi_C$. Since both B and C are representation finite and accessible, the claim follows by induction.