GORENSTEIN HOMOLOGICAL ALGEBRA

BASED ON THE TALK BY PU ZHANG

Definition.

Let $\mathscr{X}$ be a full subcategory of $\operatorname{Mod} R$ for a ring R. An exact sequence ε is called $\operatorname{Hom}_R(\mathscr{X}, -)$-exact ($\operatorname{Hom}_R(-, \mathscr{X})$-exact, respectively) if the sequence $\operatorname{Hom}_R(\varepsilon, X)$ ($\operatorname{Hom}_R(X, \varepsilon)$, respectively) is exact for each $X \in \mathscr{X}$.

Definition.

Let R be a ring. Every $\operatorname{Hom}_R(-, \operatorname{Proj} R)$-exact sequence

$$P^\circ : \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$$

of projective R-modules is called a complete projective resolution.

Definition.

Let R be a ring. An R-module M is called a Gorenstein projective module if there exists a complete projective resolution P° such that $\operatorname{Im} d_P^{-1} = M$.

Notation.

For a ring R we denote by $\operatorname{GProj} R$ the full subcategory of $\operatorname{Mod} R$ formed by the Gorenstein projective modules.

Fact.

If A is selfinjective, then $\operatorname{GProj} A = \operatorname{Mod} A$.

Fact.

Let R be a ring. If P° is a complete projective resolution, then $\operatorname{Im} d_P^i \in \operatorname{GProj} R$ for each $i \in \mathbb{Z}$. Moreover, for each $i \in \mathbb{Z}$ the sequences

$$0 \to \operatorname{Im} d_P^i \to P^{i+1} \to P^{i+2} \to \cdots$$

and

$$\cdots \to P^{i-1} \to P^i \to \operatorname{Im} d_P^i \to 0$$

are $\operatorname{Hom}_R(-, \operatorname{Proj} R)$-exact. Finally, for all $i, j \in \mathbb{Z}$, such that $i \leq j$, the sequence

$$0 \to \operatorname{Im} d_P^i \to P^{i+1} \to \cdots \to P^j \to \operatorname{Im} d_P^j \to 0$$

are $\operatorname{Hom}_R(-, \operatorname{Proj} R)$-exact.

Date: 16.07.2008.

Notation.

Let R be a ring. For a subcategory $\mathscr{X}$ of $\operatorname{Mod} R$ we put

$$^{\perp}\mathscr{X} := \{M \in \operatorname{Mod} R \mid \operatorname{Ext}^i_R(M, X) = 0 \text{ for each } M \in \mathscr{X} \text{ and } i \in \mathbb{N}_+\}.$$

Similarly we define $\mathscr{X}^{\perp}$.

Fact.

Let R be a ring. If $M \in \operatorname{GProj} R$, then $M \in {}^{\perp}\mathscr{P}$, where $\mathscr{P}$ denotes the full category of $\operatorname{Mod} R$ formed by the modules of finite projective dimension.

Fact.

Let $M \in \operatorname{Mod} R$ for a ring R. Then $M \in \operatorname{GProj} R$ if and only if $M \in {}^{\perp}(\operatorname{Proj} R)$ and M has a right projective resolution which is $\operatorname{Hom}_R(-, \operatorname{Proj} R)$-exact.

Fact.

If $M \in \operatorname{GProj} R$ for a ring R, then $\operatorname{pd}_R M \in \{0, \infty\}$.

Fact.

The category $\operatorname{GProj} R$ is closed under extensions, the kernels of epimorphism, direct summands, and (arbitrary) direct sums.

Definition.

Let R be a ring. By a proper Gorenstein projective resolution of $M \in \operatorname{Mod} R$ we mean a $\operatorname{Hom}_R(\operatorname{GProj} R, -)$-exact sequence

$$\cdots \to G_1 \to G_0 \to M \to 0$$

such that $G_i \in \operatorname{GProj} R$ for each $i \in \mathbb{N}$.

Definition.

Let R a ring. We say that $M \in \operatorname{Mod} R$ has a finite Gorenstein projective dimension if there exists an exact sequence

$$0 \to G_n \cdots \to G_0 \to M \to 0$$

such that $G_0, \ldots, G_n \in \operatorname{GProj} R$.

Theorem (Auslander/Buchweitz).

Let R be a ring and $M \in \operatorname{Mod} R$. If M has a finite Gorenstein projective dimension, then M has a proper Gorenstein projective resolution.

Theorem (Jørgensen).

If A is finite dimensional k-algebra, then every module has a proper Gorenstein projective resolution.

Definition.

Let $n \in \mathbb{N}$. A ring R is called n-Gorenstein if it is left and right noetherian, and the injective dimension of R (both as a left and a right module) is at most n.

Theorem (Iwanaga).

Let R be an n-Gorenstein ring. The following conditions are equivalent for an R-module M:

(1) $\operatorname{id} M < \infty$.
(2) $\operatorname{id} M \leq n$.
(3) $\operatorname{pd} M < \infty$.
(4) $\operatorname{pd} M \leq n$.
(5) The flat dimension of M is finite.
(6) The flat dimension of M is at most n.

Theorem.

If R is an n-Gorenstein ring, then $\operatorname{GProj} R = {}^{\perp}(\operatorname{Proj} R)$.

Theorem.

If R is an n-Gorenstein ring, then every $M \in \operatorname{Mod} R$ has a finite Gorenstein projective dimension. Consequently, every $M \in \operatorname{Mod} R$ has a proper Gorenstein projective resolution.

Theorem (Chin/Zhang).

If A is a Gorenstein algebra and T is a generalized tilting module, then the category $\mathscr{D}^b(\operatorname{mod} A)/\mathscr{K}^b(\operatorname{add} T)$ is equivalent to ${}^{\perp}T \cap T^{\perp}$ modulo the ideal of maps which factors through $\operatorname{add} T$.