AUSLANDER-REITEN COMPONENTS CONTAINING RELATIVELY PROJECTIVE MODULES

BASED ON THE TALK BY ROLF FARNSTEINER

Assumption.

Throughout the talk k will be an algebraically closed field of positive characteristic p and G a finite group. Moreover, $\mathfrak{A}_p := k[T]/T^p$.

Definition.

By a p-point we mean every algebra homomorphism $\alpha : \mathfrak{A}_p \to kG$ such that

(1) $\alpha^*(kG)$ is projective, where $\alpha^* : \operatorname{mod} kG \to \operatorname{mod} \mathfrak{A}_p$ is the functor induced by α,
(2) there exists an abelian p-subgroup P of G such that $\operatorname{Im} \alpha \subseteq kP$.

We call two p-points α and β equivalent if $\alpha^*(M)$ is projective if and only if $\beta^*(M)$ is projective for each kG-module M. By $P(G)$ we denote the space of the equivalence classes of p-points. For $M \in \operatorname{mod} kG$ we define the p-support $P(G)_M$ of M by

$$P(G)_M := \{[\alpha] \in P(G) \mid \alpha^*(M) \text{ is not projective}\}.$$

If M is a kG-module, α is a p-point, and $i \in [1, p]$, then by $\alpha_i(M)$ we denote the multiplicity of the i-dimensional indecomposable $\mathfrak{A}_p$-module $[i]$ as a direct summand of $\alpha^*(M)$.

Lemma.

If α is a p-point and $i \in [1, p-1]$, then $\alpha_i(\tau_G M) = \alpha_i(M)$ for each $M \in \operatorname{mod} kG$.

Proof.

Since kG is a symmetric algebra, $\tau_G = \Omega_G^2$. Moreover, $\Omega_{\mathfrak{A}_p}^2$ is the identity on the non-projective $\mathfrak{A}_p$-modules. Finally, if P is a minimal projective resolution of $M \in \operatorname{mod} kG$, then $\alpha^*(P)$ is a projective resolution of $\alpha^*(M)$. Consequently, $\Omega_{\mathfrak{A}_p}^n(\alpha^*(M)) \simeq \alpha^*(\Omega_G^n(M))$ for each $M \in \operatorname{mod} kG$ and $n \in \mathbb{N}$, hence the claim follows.

Definition.

Let α be a p-point. We say that $M \in \operatorname{mod} kG$ is relatively α-projective if M is a direct summand of $kG \otimes_{\mathfrak{A}_p} \alpha^*(M)$.

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Notation.

For a non-projective indecomposable kG-module M we denote by

$$\mathscr{E}_M : 0 \to \tau_G M \to E_M \xrightarrow{\pi_M} M \to 0$$

the almost split sequence terminating at M.

Lemma.

Let α be a p-point and M a non-projective indecomposable kG-module.

(1) M is relatively α-projective if and only if M is a direct summand of $kG \otimes_{\mathfrak{A}_p} [i]$ for some $i \in [1, p-1]$ such that $\alpha_i(M) \neq 0$.
(2) If M is relatively α-projective, then $P(M)_G = \{[\alpha]\}$.
(3) M is relatively α-projective if and only if $\alpha^*(\mathscr{E}_M)$ does not split.

Proof.

(3) Assume that M is not relatively α-projective. We show that $\alpha^*(\mathscr{E}_M)$ splits. Let $\mu : kG \otimes_{\mathfrak{A}_p} \alpha^*(M) \to M$ be the canonical map: $\mu(a \otimes m) := am$ for $a \in kG$ and $m \in M$. Since M is not relatively α-projective, there exists $\omega : kG \otimes_{\mathfrak{A}_p} \alpha^*(M) \to E_M$ such that $\mu = \pi_M \circ \omega$. By the isomorphism

$$\mathrm{Hom}_{kG}(kG \otimes_{\mathfrak{A}_p} \alpha^*(M), Z) \simeq \mathrm{Hom}_{\mathfrak{A}_p}(\alpha^*(M), \alpha^*(Z))$$

functorial in $Z \in \operatorname{mod} kG$, there exists a map $\tilde{\omega} : \alpha^*(M) \to \alpha^*(E_M)$ such that $\alpha^*(\pi_M) \circ \tilde{\omega} = \mathrm{Id}_{\alpha^*(M)}$, and the claim follows.

Definition.

A component Θ of $\Gamma_s(G)$ is called α-split for a p-point α if $\alpha^*(\mathscr{E}_M)$ splits for all $M \in \Theta$. A component Θ of $\Gamma_s(G)$ is called p-split if Θ is α-split for each p-point α.

Proposition.

Let Θ be a component of $\Gamma_s(G)$ and α a p-point. If Θ is not α-split, then Θ is either finite or a tube.

Proposition.

Let α be a p-point and Θ be a tube in $\Gamma_s(G)$ such that $\mathrm{ql}(N) = 1$ for each $N \in \Theta$ which is relative α-projective. Then there exist $n, n_1, \dots, n_p \in \mathbb{N}$ such that

$$\alpha_i(X) = \Big(n - \sum_{j \in [1,p-1]} a_{i,j} n_j\Big)\,\mathrm{ql}(X) + \sum_{j \in [1,p-1]} a_{i,j} n_j$$

for each $X \in \Theta$, where $(a_{i,j})$ is the Cartan martix of type A_{p-1}.