“DUAL” GARSIDE STRUCTURES FOR FINITE-TYPE ARTIN GROUPS VIA QUIVER

BASED ON THE TALK BY HUGH THOMAS

Definition.

Let M be a monoid generated by elements Σ and relations R. If $g, h \in M$, then we write $h <_l g$ if there exist $p_1, \dots, p_r \in \Sigma$ and $s \in [0, r]$ such that the following conditions are satisfied:

(1) $g = p_1 \cdots p_r$ and $h = p_1 \cdots p_s$,
(2) $r = \ell(g)$ and $s = \ell(h)$, where for an element $k \in M$ we put

$$\ell(k) := \min\{t \in \mathbb{N} \mid k = q_1 \cdots q_t \text{ for } q_1, \dots, q_t \in \Sigma\}.$$

For $g \in M$ we denote by $L(g)$ the set of $h \in M$ such that $h <_l g$. Similarly, we define the relation $<_r$ and the set $R(g)$ for $g \in M$.

By a Garside element for (Σ, R) we mean $\Delta \in M$ such that $L(\Delta) = R(\Delta)$, $p <_l \Delta$ for each $p \in \Sigma$, $L(\Delta)$ form a lattice under $<_l$, and $R(\Delta)$ form a lattice under $<_r$.

Definition.

Let $\mathbf{m} = (m_{i,j} \mid i, j \in [1, n],\ i < j)$ for some $n \in \mathbb{N}_+$ be such that $m_{i,j} \in [2, \infty)$ for each $i, j \in [1, n]$, $i < j$. We define $W = W(\mathbf{m})$ as the group generated by $s_1, \dots, s_n$ and relations $s_i^2 = e$, $i \in [1, n]$, and

$$\underbrace{s_i s_j s_i \cdots}_{m_{i,j}} = \underbrace{s_j s_i s_j \cdots}_{m_{i,j}}$$

for each $i, j \in [2, \infty)$, $i < j$, such that $m_{i,j} < \infty$ (we call the latter relations braid relations). The groups of this form are called Weyl groups. Similarly, we denote by $A = A(\mathbf{m})$ the group generated by elements $\sigma_1, \dots, \sigma_n$ and the braid relations, and call it an Artin group. We say that an artin group is of finite type if the corresponding Weyl group is finite.

Examples of Artin groups of finite type are provided by the groups associated with simply-laced Dynkin diagrams. Namely, let Q be a simply-laced Dynkin diagram. For simplicity we always assume that $Q_0 = [1, n]$, where n is the number of vertices of Q. We put

$$m_{i,j} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected by an edge,} \\ 3 & \text{if } i \text{ and } j \text{ are connected by an edge,} \end{cases}$$

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and denote the corresponding Artin group by $A(Q)$. Moreover, we put $\Sigma(Q) := \{\sigma_i \mid i \in [1,n]\}$, denote by $R(Q)$ the set of the corresponding braid relations, and by $W(Q)$ the corresponding Weyl group.

THEOREM.

Let Q be a simply-laced Dynkin diagram. If $s_{i_1} \cdots s_{i_N}$ is a reduced presentation of the longest element of $W(Q)$, then $\sigma_{i_1} \cdots \sigma_{i_N}$ is a Garside element for $(\Sigma(Q), R(Q))$.

DEFINITION.

Let Γ be a simply-laced Dynkin quiver with n vertices. Moreover, assume that $i < j$ if there is an arrow $i \to j$, for each $i, j \in [1,n]$.

A representation E of Γ is called exceptional if E is indecomposable and $\mathrm{Ext}^1_\Gamma(E,E) = 0$. A sequence $(E_1, \dots, E_r)$ of exceptional representations of Γ is called exceptional if $\mathrm{Hom}_\Gamma(E_j, E_i) = 0 = \mathrm{Ext}^1_\Gamma(E_j, E_i)$ for each $i, j \in [1,n]$ such that $i < j$.

If $(E_1, \dots, E_r)$ is an exceptional sequence, then $r \leq n$. Moreover, there exist exceptional representations $E_{r+1}, \dots, E_n$ of Γ such that $(E_1, \dots, E_n)$ is an exceptional sequence. Let $\mathscr{E}_r$ denotes the set of exceptional sequences of length r for $r \in [1,n]$. If $i \in [1, n-1]$ and $(E_1, \dots, E_n)$ is an exceptional sequence, then there exists a unique exceptional object E such that $(E_1, \dots, E_{i-1}, E, E_i, E_{i+2}, \dots, E_n)$ is an exceptional sequence. In this way we obtain an action of the braid group B_r on $\mathscr{E}_r$. The B_r-orbits under this action correspond to the exact abelian extension closed subcategories of $\operatorname{rep}\Gamma$. Moreover, the action of B_n on $\mathscr{E}_n$ is transitive.

Put

$$\Sigma(\Gamma) := \{\theta_E \mid E \text{ is an exceptional representation of } \Gamma\}$$

and

$$R(\Gamma) := \{\theta_{E_1} \cdots \theta_{E_n} = \theta_{F_1} \cdots \theta_{F_n} \mid (E_1, \dots, E_n), (F_1, \dots, F_n) \in \mathscr{E}_n\}.$$

Let $D(\Gamma)$ be the group generated by $\Sigma(\Gamma)$ and relations $R(\Gamma)$.

THEOREM.

Let Γ be a simply-laced Dynkin quiver with n vertices. Then $\Delta = \theta_{E_1} \cdots \theta_{E_n}$ is a Garside element for $(\Sigma(\Gamma), R(\Gamma))$.

PROOF.

Let $\Delta := \theta_{E_1} \cdots \theta_{E_n}$. Then $L(\Delta)$ is given by the B_r-orbits in $\mathscr{E}_r$ and its lattice structure is given by the intersection of the corresponding subcategories of $\operatorname{rep}\Gamma$.

THEOREM.

Let Γ be a simply-laced Dynkin quiver with the underlying diagram Q. Then $A(Q) \simeq D(\Gamma)$.

PROOF.

The isomorphism is given by $\sigma_i \mapsto \theta_{E_i}$, $i \in [1, n]$, where n is the number of vertices of Q.

COROLLARY.

Let Q be a simply-laced Dynkin diagram with n vertices. If T is the set of reflections in $W(Q)$ and $c := s_1 \cdots s_n$, where $s_1, \ldots, s_n$ are the simple reflections, then $A(Q)$ is isomorphic generated by θ_t, $t \in T$, and the relations $\theta_{t_1} \cdots \theta_{t_n} = \theta_{s_1} \cdots \theta_{s_n}$ for $t_1, \ldots, t_n \in T$ such that $t_1 \cdots t_n = c$. Moreover, $\theta_{s_1} \cdots \theta_{s_n}$ is a Garside element.

PROOF.

Let Γ be a simply-laced Dynkin quiver with the underlying diagram Q such that for each arrow $i \to j$ in Γ, $i < j$. It follows that the reflections in $W(Q)$ correspond to the exceptional representations of Q and the factorizations of c are given by exceptional sequences, hence the claim follows from the above theorems.