ON TRIANGULATED CATEGORIES AND ENVELOPING ALGEBRAS

BASED ON THE TALK BY FAN XU

Assumption.

Throughout the talk $\mathbb{F}_q$ is a fixed finite field. All considered categories are additive over $\mathbb{F}_q$.

Definition.

An abelian category $\mathscr{A}$ is called finitary if $|\operatorname{Ext}^n_{\mathscr{A}}(X,Y)| < \infty$ for all objects X and Y of $\mathscr{A}$. Similarly, a triangulated category $\mathscr{T}$ is called finitary if $|\operatorname{Hom}_{\mathscr{T}}(X,Y)| < \infty$ for all objects X and Y of $\mathscr{T}$.

Definition.

Let $\mathscr{A}$ be a small finitary abelian category. For objects X, Y, and L of $\mathscr{A}$ we put

$$W(X,Y;L) := \{(f,g) \mid 0 \to X \xrightarrow{f} L \xrightarrow{g} Y \to 0\}.$$

Then $\operatorname{Aut}(X) \times \operatorname{Aut}(Y)$ acts on $W(X,Y;L)$ and we denote by $F^L_{X,Y}$ the number of orbits with respect to this action. Let $\mathscr{H}$ be the $\mathbb{Q}$-vector space whose basis is formed by the isoclasses of the objects of $\mathscr{A}$. We define the multiplication in $\mathscr{H}$ by

$$[Y] \cdot [X] := \sum_{[L]} F^L_{X,Y} [L].$$

Since

$$\sum_{[L]} F^L_{X,Y} \cdot F^M_{Z,L} = \sum_{[L]} F^M_{L,Y} \cdot F^L_{Z,X}$$

for each object M of $\mathscr{A}$, this multiplication is associative.

Definition.

Let $\mathscr{T}$ be a finitary Krull–Schmidt triangulated category such that $\operatorname{Hom}_{\mathscr{T}}(X[i],Y) = 0$ for all objects X and Y of $\mathscr{T}$ and $i \gg 0$. For objects X, Y, and L of $\mathscr{T}$ we denote by $\operatorname{Hom}_{\mathscr{T}}(X,L)_Y$ the set of $f \in \operatorname{Hom}_{\mathscr{T}}(X,L)$ such that the cone of f is isomorphic to Y, and we put

$$g^L_{X,Y} := \frac{|\operatorname{Hom}_{\mathscr{T}}(X,L)_Y|}{|\operatorname{Aut}(X)|} \cdot \frac{\prod_{i \in \mathbb{N}_+} |\operatorname{Hom}_{\mathscr{T}}(X[i],L)|^{(-1)^i}}{\prod_{i \in \mathbb{N}_+} |\operatorname{Hom}_{\mathscr{T}}(X[i],X)|^{(-1)^i}}$$

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(the above formula is called Toen formula). Let $\mathscr{H}$ be the $\mathbb{Q}$-vector space whose basis is formed by the isoclasses of the objects of $\mathscr{T}$. We define the multiplication in $\mathscr{H}$ by

$$[Y]\cdot[X] := \sum_{[L]} g^L_{X,Y}[L].$$

Since

$$\sum_{[L]} g^L_{X,Y}\cdot g^M_{Z,L} = \sum_{[L]} g^M_{L,Y} g^L_{Z,X}$$

for each object M of $\mathscr{T}$, this multiplication is associative.

Definition.

Let $\mathscr{T}$ be a finitary Krull–Schmidt triangulated category such that $T^2 = \mathrm{Id}$. For objects X, Y, and L of $\mathscr{A}$ we put

$$W(X,Y;L) := \{(f,g) \mid 0 \to X \xrightarrow{f} L \xrightarrow{g} Y \to 0\}.$$

Then $\mathrm{Aut}(X)\times\mathrm{Aut}(Y)$ acts on $W(X,Y;L)$ and we denote by $F^L_{X,Y}$ the number of orbits with respect to this action. Let $\mathscr{H}$ be the free $\mathbb{Z}/(q-1)$-module space whose basis is formed by the isoclasses of the objects of $\mathscr{T}$, and let $\mathscr{H}'$ be the direct sum of $\mathscr{H}$ and the Grothendieck group of $\mathscr{T}$ tensored with $\mathbb{Z}/(q-1)$. Peng and Xiao defined the Lie bracket in $\mathscr{H}'$ by

$$[[X],[Y]] := \begin{cases} \sum_{[L]}(F^L_{Y,X} - F^L_{X,Y})[L] & X \not\simeq Y[1], \\ \frac{\mathbf{dim}\, X}{d(X)} & X \simeq Y[1], \end{cases}$$

where $d(X)$ is the dimension of $\mathrm{End}_{\mathscr{T}}(X)/\operatorname{rad}\mathrm{End}_{\mathscr{T}}(X)$. The above Lie bracket satisfies Jacobi identity. One may also define

$$g^L_{X,Y} := \begin{cases} \frac{|\mathrm{Hom}_{\mathscr{T}}(X,L)_Y|}{|\mathrm{Aut}(X)|} & X \not\simeq L\oplus Y[1], \\ |\mathrm{Hom}_{\mathscr{T}}(L,Y[1])|\frac{|\mathrm{Hom}(X,L)_Y|}{|\mathrm{Aut}(X)|} & X \simeq L\oplus Y[1] \end{cases}$$

for objects X, Y, and L of $\mathscr{T}$. Then in $\mathbb{Z}[\frac{1}{q}]/(q-1) \simeq \mathbb{Z}/(q-1)$

$$\sum_{[L]} g^L_{X,Y}\cdot g^M_{Z,L} = \sum_{[L]} g^M_{L,Y}\cdot g^L_{Z,X}$$

for each object M of $\mathscr{T}$.