INTRODUCTION TO SCHUR ALGEBRAS. I

BASED ON THE TALK BY STEPHEN DOTY

Throughout the talk we assume that $n, r \in \mathbb{N}_+$.

If R is a (commutative) ring with identity, then the symmetric group $\mathfrak{S}_r$ acts (on the right) on $(R^n)^{\otimes r}$ by place permutation. By the Schur algebra $S_R(n,r)$ we mean the algebra of $\mathfrak{S}_r$-endomorphisms of $(R^n)^{\otimes r}$. Observe that $S_R(n,r) = \operatorname{End}_R((R^n)^{\otimes r})$ if either $n = 1$ or $r = 1$. Moreover,

$$S_R(2,2) \simeq \{A \in \mathbb{M}_4(R) \mid a_{2,2} = a_{3,3},\ a_{2,3} = a_{3,2},\\ a_{1,2} = a_{1,3},\ a_{4,2} = a_{4,3},\ a_{2,1} = a_{3,1},\ a_{2,4} = a_{3,4}\}.$$

One may show that $S_R(n,r) \simeq R \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,r)$.

From now on we assume that K is an infinite field.

Recall that $\mathrm{GL}_n(K)$ acts (on the left) on K^n by matrix multiplication. Consequently, $\mathrm{GL}_n(K)$ acts on $(K^n)^{\otimes r}$ diagonally. The actions of $\mathfrak{S}_r$ and $\mathrm{GL}_n(K)$ on $(K^n)^{\otimes r}$ obviously commute. They also give rise to the morphisms

$$\rho_1 : K\,\mathrm{GL}_n(K) \to \operatorname{End}_K((K^n)^{\otimes r}) \quad \text{and} \quad \rho_2 : K\mathfrak{S}_r \to \operatorname{End}_K((K^n)^{\otimes r}).$$

One may show that $\operatorname{Im}\rho_1 = S_K(n,r)$ and $\operatorname{Im}\rho_2 = \operatorname{End}_{\mathrm{GL}_n(K)}((K^n)^{\otimes r})$. The same theorem is obtained if we replace $K\,\mathrm{GL}_n(K)$ by $K\,\mathrm{SL}_n(K)$ or its algebra $\mathscr{U}_K$ of distributions. Recall, that $\mathscr{U}_K := K \otimes_{\mathbb{Z}} \mathscr{U}_{\mathbb{Z}}$, where $\mathscr{U}_{\mathbb{Z}}$ is the Konstant $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{gl}_n$.

Let $A_K(n)$ be the algebra of polynomial functions of $\mathrm{GL}_n(K)$ and $A_K(n,r)$ be the space of homogeneous polynomials of total degree r. Observe that $A_K(n)$ is a bialgebra with the comultiplication Δ given by

$$\Delta X_{i,j} := \sum_{k \in [1,n]} X_{i,k} \otimes X_{k,j}.$$

It easily follows that $A_K(n,r)$ is a subcoalgebra of $A_K(n)$. Consequently, the linear dual $A_K(n,r)^*$ of $A_K(n,r)$ is an algebra. Indeed, if A is coalgebra with a comultiplication Δ, then the formula $f \cdot g := m_K \circ (f \otimes g) \circ \Delta$ defines a multiplication in A^*, where $m_K : K \otimes K \to K$ is the multiplication. It can be shown that $S_K(n,r) \simeq A_K(n,r)^*$. In particular,

$$\dim_K S_K(n,r) = \binom{n^2 + r - 1}{n^2 - 1} = \binom{n^2 + r - 1}{r}.$$

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THEOREM (DOTY/GIAQUINTO).

Let $H_1, \ldots, H_n$ be the standard basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_n$. Then the kernel of the map $\mathscr{U}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}((\mathbb{C}^n)^{\otimes r})$ is generated by

$$H_i(H_i - 1) \cdots (H_i - r),\ i \in [1, n],$$

and

$$H_1 + \cdots + H_n - r.$$

Schur proved that every polynomial representation of GL_n is a direct sum of homogeneous representation. Moreover, one may prove that the category of homogeneous representation of GL_n of degree r is equivalent to the category of $S_K(n, r)$-modules. Finally, we remark that $\mathrm{Hom}_{S_K(n,r)}((K^n)^{\otimes r}, -)$ is an exact functor from $\operatorname{mod} S_K(n, r)$ to $\operatorname{mod} K\mathfrak{S}_r$.