THE COMPOSITION ALGEBRA AT $q = 0$ AND THE COMPOSITION MONOID

BASED ON THE TALK BY STEFAN WOLF

Throughout the talk Q is a fixed finite connected quiver and k a fixed algebraically closed field. For a field K and $i \in Q_0$, we denote by $S_i^{(K)}$ the corresponding simple K-representation of Q.

For $\mathbf{d} \in \mathbb{N}^{Q_0}$ we denote by $\operatorname{Rep}_Q(\mathbf{d})$ the variety of k-representations of Q of dimension vector $\mathbf{d}$. Recall that we have a natural action of $\operatorname{GL}(\mathbf{d}) := \prod_{i \in Q_0} \operatorname{GL}(d_i)$ on $\operatorname{Rep}_Q(\mathbf{d})$. We say that $\mathscr{A} \subset \operatorname{Rep}_Q(\mathbf{d})$ is stable if $\operatorname{GL}(\mathbf{d}) \cdot \mathscr{A} \subset \mathscr{A}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be closed irreducible stable subsets of $\operatorname{Rep}_Q(\mathbf{d})$ and $\operatorname{Rep}_Q(\mathbf{e})$, respectively, for $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0}$. We put

$$\begin{aligned} \mathscr{A} \cdot \mathscr{B} := \{X \in \operatorname{Rep}_Q(\mathbf{d} + \mathbf{e}) \mid & \text{ there exists an exact sequence} \\ & 0 \to B \to X \to A \to 0 \text{ with } B \in \mathscr{B} \text{ and } A \in \mathscr{A}\}. \end{aligned}$$

Reineke proved that $\mathscr{A} \cdot \mathscr{B}$ is a closed irreducible stable subset of $\operatorname{Rep}_Q(\mathbf{d} + \mathbf{e})$. Consequently, $\cdot$ gives an associative multiplication in the set of closed irreducible stable subsets of the varieties $\operatorname{Rep}_Q(\mathbf{d})$, $\mathbf{d} \in \mathbb{N}^{Q_0}$. The monoid obtained in this way is called the monoid of generic extensions and denoted $\mathscr{M}_k(Q)$. By $\mathscr{C}_k(Q)$ we denote the submonoid of $\mathscr{M}_k(Q)$ generated by the orbits of $S_i^{(k)}$, $i \in Q_0$, and call it the composition monoid of Q.

Let F be a finite field. By $\operatorname{mod}_F Q$ we denote the chosen set of representatives of the isomorphism classes of F-representations of Q. The Hall algebra $\mathscr{H}_F(Q)$ is a $\mathbb{Q}$-algebra with basis formed by the elements of $\operatorname{mod}_F Q$ and with the multiplication defined by $M \cdot N := \sum_{X \in \operatorname{mod}_F Q} F_{M,N}^X X$, where

$$F_{M,N}^X := \#\{U \subset X \mid U \simeq N,\ X/U \simeq M\}.$$

The composition algebra $\mathscr{C}_F(Q)$ is the subalgebra of $\mathscr{H}_F(Q)$ generated by $S_i^{(F)}$, $i \in Q_0$. There exists a $\mathbb{Q}[q]$-algebra $\mathscr{C}_q(Q)$ such that

$$\mathscr{C}_F(Q) = \mathscr{C}_q(Q) \otimes_{\mathbb{Q}[q]} \mathbb{Q}[q]/(q - |F|)$$

for each finite field F. Moreover, there exists elements u_i, $i \in Q_0$, such that u_i is mapped to $S_i^{(F)}$ for each $i \in Q_0$ and each finite field F.

We define

$$\mathscr{C}_0(Q) := \mathscr{C}_q(Q) \otimes_{\mathbb{Q}[q]} \mathbb{Q}[q]/(q).$$

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For $i \in Q_0$ we denote the image of u_i in $\mathscr{C}_0(Q)$ also by u_i. Moreover, if there are no oriented cycles in Q, then we assume that $Q_0 = [1, n]$ and $j > i$ provided there is an arrow $j \to i$. In the above situation we put $u_{\mathbf{d}} := u_n^{d_n} \cdots u_1^{d_1}$ for $\mathbf{d} \in \mathbb{N}^n$.

THEOREM.

Let Q be a quiver which is either Dynkin or extended Dynkin. Then there exists an epimorphism of algebras $\Psi : \mathscr{C}_0(Q) \to \mathbb{Q}\mathscr{C}_k(Q)$ such that $\Psi(u_i)$ is the orbit of $S_i^{(k)}$ for each $i \in Q_0$. Moreover, $\operatorname{Ker} \Psi = 0$ if Q is either Dynkin or an oriented cycle, and $\operatorname{Ker} \Psi$ is generated by $(u_\delta)^r - u_{r\delta}$, $r \in \mathbb{N}_+$, where δ is an isotropic root of Q.