INTRODUCTION TO SCHUR ALGEBRAS. II

BASED ON THE TALK BY STEPHEN DOTY

Throughout the talk K will be a field, G a reductive connected algebraic group, and T a maximal torus in G.

Recall that the simple G-modules are parameterized by the set $X(T)^+$ of dominant weights.

Let π be a set of dominant weights. Let $\mathscr{C}(\pi)$ be the full subcategory of the category of G-modules formed by the G-modules M such that each composition factor of M has the highest weight in π. For a G-module M we denote by $\mathscr{O}_\pi M$ the largest submodule of M which belongs to $\mathscr{C}(\pi)$. In this way we obtain a left exact functor $\mathscr{O}_\pi$ from the category of G-modules to $\mathscr{C}(\pi)$, which is called a truncation functor.

A set π of dominant weights is called saturated if for all $\lambda, \mu \in X(T)^+$ such that $\lambda \leq \mu$ and $\mu \in \pi$ (where $\leq$ is the dominace order) we have $\lambda \in \pi$.

Proposition.
Let π be a saturated subset of $X(T)^+$. Then $\mathscr{C}(\pi)$ is a highest weight category in the sense of Cline/Parhsall/Scott and

$$\mathrm{Ext}^\circ_{\mathscr{C}(\pi)}(M, N) \simeq \mathrm{Ext}^\circ_G(M, N)$$

for all $M, N \in \mathscr{C}(\pi)$.

Let π be a finite and saturated set of dominant weights. We put $A_K(\pi) := \mathscr{O}_\pi K[G]$ and $S_K(\pi) := A_K(\pi)^*$, and call $S_K(\pi)$ a generalized Schur algebra. One shows that $S_K(\pi)$ is a finite dimensional quasi-herediatry algebra. Moreover, it can be proved that $\mathscr{C}(\pi)$ is equivalent to the category of $S_K(\pi)$-modules,

$$\dim_K S_K(\pi) = \sum_{\lambda \in \pi} \dim_K^2 \Delta(\lambda),$$

where for $\lambda \in \pi$ we denote by $\Delta(\lambda)$ the Weyl module of highest weight λ. Finally, there exists the integral form $S_{\mathbb{Z}}(\pi)$ of $S_K(\pi)$ independent on K such that $S_K(\pi) \simeq K \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(\pi)$.

Donkin proved that if $G = \mathrm{GL}_n$ for $n \in \mathbb{N}_+$ and π is the set of dominant weights in $(K^n)^{\otimes r}$ for $r \in \mathbb{N}_+$, then $S_K(\pi) \simeq S_K(n, r)$.

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Let R be a commutative ring and fix $v \in R^\times$. Put $q := v^2$. For $r \in \mathbb{N}_+$ we define $\mathbb{H} := \mathbb{H}_q(\mathfrak{S}_r)$ as the associative R-algebra generated by T_1, …, T_{r-1} and the following relations:

(1) $(T_i + 1)(T_i - q) = 0$, $i \in [1, r-1]$,
(2) $T_iT_jT_i = T_jT_iT_j$, $i, j \in [1, r-1]$, $|i - j| = 1$,
(3) $T_iT_j = T_jT_i$, $i, j \in [1, r-1]$, $|i - j| > 1$.

Relations (2) and (3) are called braid relations. Some authors replace relation (1) by the relation $(T_i - v)(T_i + v^{-1}) = 0$, $i \in [1, n-1]$. This leads to an equivalent theory. Observe that $\mathbb{H} = RS_r$ if $q = 1$. For a sequence $I = (i_1, \ldots, i_r)$ with $i_1, \ldots, i_r \in [1, n]$ we put $X_I := x_{i_1} \otimes \cdots \otimes x_{i_r}$, where x_1, …, x_n is the standard basis of R^n. The we define the (right) action of $\mathbb{H}$ on $(R^n)^{\otimes r}$ by

$$X_I T_k := \begin{cases} v^2 X_I & i_k = i_{k+1}, \\ v X_{I(k,k+1)} & i_k < i_{k+1}, \\ v X_{I(k,k+1)} + (v^2 - 1) X_I & i_k > i_{k+1}, \end{cases}$$

for a sequence $I = (i_1, \ldots, i_r)$ with $i_1, \ldots, i_r \in [1, n]$ and $k \in [1, r-1]$, where $I(k, k+1) := (j_1, \ldots, j_r)$ and

$$j_l := \begin{cases} i_l & l \neq k, k+1, \\ i_{k+1} & l = k, \\ i_k & l = k+1, \end{cases}$$

for $l \in [1, r]$. We define $\mathscr{S}_q(n, r) := \mathrm{End}_{\mathbb{H}}((R^n)^{\otimes r})$ and call it a Dipper–James q-Schur algebra.