A COUNTEREXAMPLE TO THE TELESCOPE CONJECTURE OF GLOBAL DIMENSION 2

BASED ON THE TALK BY JAN ŠŤOVÍČEK

§1. Telescope Conjecture

Throughout this section Λ is a ring.

Definition.
A functor $L : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)$ is called a *localization functor* if there exists a natural transformation $\eta : \operatorname{Id}_{\mathscr{D}(\operatorname{Mod}\Lambda)} \to L$ such that $L(\eta_X) = \eta_{LX}$ and η_{LX} is an isomorphism for each complex X of Λ-modules.

Notation.
For a functor $L : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)$ we put

$$\operatorname{Ker} L := \{X \in \mathscr{D}(\operatorname{Mod}\Lambda) \mid LX = 0\}.$$

Fact.
If $L : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)$ is a localization functor, then $\operatorname{Ker} L$ is a localizing class.

Fact.
Let $L : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)$ be a localization functor and $Q : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)/\operatorname{Ker} L$ the quotient functor. Then there exists an equivalence $F : \mathscr{D}(\operatorname{Mod}\Lambda)/\operatorname{Ker} L \to \operatorname{Im} L$ such that $F \circ Q = L$. Moreover, the inclusion functor $\operatorname{Im} L \to \mathscr{D}(\operatorname{Mod}\Lambda)$ is right adjoint to L.

Notation.
If $\mathscr{L}$ is a localizing class in $\mathscr{D}(\operatorname{Mod}\Lambda)$, such that the quotient functor $Q : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)/\mathscr{L}$ has a right adjoint R, then we put $L_{\mathscr{L}} := R \circ Q$.

Fact.
Let $\mathscr{L}$ be a localizing class in $\mathscr{D}(\operatorname{Mod}\Lambda)$. If the quotient functor $Q : \mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)/\mathscr{L}$ has a right adjoint R, then R is fully faithful and $L_{\mathscr{L}}$ is a localization functor.

Definition.
A localization functor $\mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)$ is called smashing if it preserves coproducts.

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Notation.

For a set $\mathscr{S}$ of perfect complexes we denote by $\mathscr{L}(\mathscr{S})$ the localizing class of $\mathscr{D}(\operatorname{Mod} R)$ generated by $\mathscr{S}$.

Fact.

If $\mathscr{S}$ is a set of perfect complexes, then $L_{\mathscr{L}(\mathscr{S})}$ is a smashing localizing functor.

Conjecture (Telescope Conjecture).

Every smashing localizing functor $\mathscr{D}(\operatorname{Mod}\Lambda) \to \mathscr{D}(\operatorname{Mod}\Lambda)$ is of the form $L_{\mathscr{L}(\mathscr{S})}$ for a set $\mathscr{S}$ of perfect complexes.

Theorem (Neeman).

If Λ is commutative noetherian, then Telescope Conjecture holds for Λ.

Theorem (Krause/Šťovíček).

If Λ is right hereditary, then Telescope Conjecture holds for Λ.

Remark.

Keller constructed a ring for which Telescope Conjecture does not hold.

Proposition (Keller).

Assume that there exists an ideal I in Λ contained in the Jacobson radical of Λ such that $\operatorname{Tor}_n^{\Lambda}(\Lambda/I, \Lambda/I) = 0$ for each $n \in \mathbb{N}_+$. Then $- \otimes_{\Lambda}^{\mathbb{L}} (\Lambda/I)$ is a smashing localization functor whose kernel contains no nonzero perfect complexes.

Proof.

Put $L := - \otimes_{\Lambda}^{\mathbb{L}} (\Lambda/I)$. One knows that L preserves coproducts. Next, one constructs a natural transformation $\eta : \operatorname{Id} \to L$ using the quotient map $\Lambda \to \Lambda/I$ and the isomorphism $\operatorname{Id} \simeq - \otimes_{\Lambda}^{\mathbb{L}} \Lambda$. Moreover, one shows that $L(\eta_X)$ is an isomorphism for each complex X. Finally, assume that $LP = 0$ for a perfect complex P. Fix n such that $P_m = 0$ for each $m \in [n+1, \infty)$. Since I is contained in the Jacobson radical of Λ it follows that d_{n-1}^P is an epimorphism, thus we prove that $P = 0$ by easy induction.

§2. An example of global dimension 2

Definition.

A commutative domain R is called a valuation domain if for all $a, b \in R$ either $a \mid b$ or $b \mid a$.

Notation.

For a valuation domain R we define its value group $G(R)$ as the quotient $Q^{\times}/U$, where Q is the quotient field of R and U is the group of units of R.

Remark.

If R is a valuation domain, then $G(R)$ is a totally ordered abelian group.

Notation.

Let k be a field and G be a totally ordered abelian group. Let Q be the quotient field of the group algebra kG of G. By R_G^k we denote the subring of Q formed by the rational functions of nonnegative degree.

Theorem.

If k be a field and G be a totally ordered abelian group, then R_G^k is a valuation domain with the residue field k and the value group G.

Theorem.

Let k be a field, $G := \mathbb{Z}^{(\mathbb{N})}$ with the lexicographic order, and $\Lambda := R_G^k$. Then $\operatorname{gl.dim} \Lambda = 2$ and Telescope Conjecture does not hold for Λ.

Proof.

If I an ideal of Λ, then I is countable generated, hence $\operatorname{pd}_\Lambda I \leq 1$ and $\operatorname{gl.dim} \Lambda \leq 2$. Now, it remains to verify Keller's Criterion for the maximal ideal of Λ.