AUSLANDER GENERATORS OF ITERATED TILTED ALGEBRAS

BASED ON THE TALK BY DIETER HAPPEL

The talk was based on a joint work with Flávio Coelho and Luise Unger.

Throughout the talk k is a fixed field. All algebras and modules are finite dimensional over k.

§1. Representation dimension

Definition.

A module M over a k-algebra Λ is called a *generator-progenerator of* $\operatorname{mod}\Lambda$ if $\Lambda \oplus D\Lambda \in \operatorname{add} M$.

Definition.

By a *representation dimension* $\operatorname{rep.dim}\Lambda$ of a finite dimensional k-algebra Λ we mean

$$\min\{\operatorname{gl.dim}\operatorname{End}_\Lambda(M) \mid M \text{ is a generator-progenerator of } \operatorname{mod}\Lambda\}.$$

Remark.

Auslander proved that $\operatorname{rep.dim}\Lambda \leq 2$ for an algebra Λ if and only if Λ is representation finite.

Remark.

Iyama proved that $\operatorname{rep.dim}\Lambda < \infty$ for each algebra Λ.

Proposition.

If there exists a generator-progenetator of $\operatorname{mod}\Lambda$ for an algebra Λ with the property that for each Λ-module X there exists an exact sequence

$$0 \to M_1 \to M_0 \to X \to 0$$

such that $M_0, M_1 \in \operatorname{add} M$ and the induced sequence

$$0 \to \operatorname{Hom}_\Lambda(M, M_1) \to \operatorname{Hom}_\Lambda(M, M_0) \to \operatorname{Hom}_\Lambda(M, X) \to 0$$

is exact, then $\operatorname{rep.dim}\Lambda \leq 3$.

Example.

If H is a hereditary algebra, then $\operatorname{rep.dim} H \leq 3$ (it is enough to take $M := \Lambda \oplus D\Lambda$).

Example.

Assem, Platzek, and Trepode proved that if Λ is a tilted algebra, then $\operatorname{rep.dim}\Lambda \leq 3$.

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Example.

Oppermann proved that if Λ is a quasitilted algebra, then $\operatorname{rep.dim} \Lambda \leq 3$.

§2. Iterated tilted algebras

Definition.

Let H be a hereditary algebra. We say that an algebra Λ is *iterated tilted of type H* if there exists algebras $\Lambda_0, \ldots, \Lambda_m$ such that $\Lambda_0 = H$, $\Lambda_m = \Lambda$, and for each $i \in [1, m]$ there exists a tilting Λ_{i-1}-module T_i such that $\Lambda_i = \operatorname{End}_{\Lambda_{i-1}} T_i$.

Example.

Let Δ be a finite oriented tree. If $\Lambda := k\Delta / \langle w \mid \ell(w) \geq 2 \rangle$, then Λ is an iterated tilted algebra of type $k\Delta$.

Definition.

Let H be a hereditary algebra. We say that an algebra Λ is *piecewise hereditary of type H* if $\mathscr{D}^b(\Lambda)$ and $\mathscr{D}^b(H)$ are equivalent as triangulated categories.

Theorem (Happel/Rickard/Schoefield).

Let H be a hereditary algebra. For an algebra Λ the following conditions are equivalent.

(1) Λ is iterated tilted of type H.

(2) Λ is piecewise hereditary of type H.

(3) There exists a tilting object T in $\mathscr{D}^b(H)$ such that $\Lambda = \operatorname{End}(T)$.

§3. Main result

Assumption.

Throughout this section we fix an iterated tilted algebra Λ. We also fix a hereditary algebra H and a tilting object T in $\mathscr{D}^b(H)$ such that $\Lambda = \operatorname{End}(T)$.

Notation.

Let $F : \mathscr{D}^b(H) \to \mathscr{D}^b(\Lambda)$ be the triangle equivalence induced by T and denote by G a quasi-inverse to F.

Notation.

If $t \in \mathbb{Z}$, then $\mathscr{H}_t := \operatorname{ind} H[t]$.

Remark.

If X is an indecomposable object of $\mathscr{D}^b(H)$, then $X \in \mathscr{H}_t$ for some t.

Assumption.

Without loss of generality we may assume that $T = \bigoplus_{t \in [0,r]} T_t[t]$ for some $r \in \mathbb{N}$ and H-modules $T_0, \ldots, T_t$.

REMARK.

If X is an indecomposable Λ-module, then $GX \in \bigcup_{t\in[0,r+1]} \mathscr{H}_t$.

NOTATION.

For $t \in [0, r+1]$ we put

$$\mathscr{U}_t := \operatorname{add}\{X \in \operatorname{ind}\Lambda \mid GX \in \mathscr{H}_t\}.$$

NOTATION.

For an H-module X we put

$$\mathscr{T}(X) := \{Y \in \operatorname{mod} H \mid \operatorname{Ext}^1_\Lambda(X, Y) = 0\},$$
$$\mathscr{F}(X) := \{Y \in \operatorname{mod} H \mid \operatorname{Hom}^1_\Lambda(X, Y) = 0\},$$

and

$$X^\perp := \mathscr{T}(X) \cap \mathscr{F}(X).$$

NOTATION.

For $t \in [0, r+1]$ we put

$$\mathscr{V}_t := \mathscr{T}(T_t) \cap \mathscr{F}(T_{t-1}) \cap \bigcap_{\substack{i\in[0,r+1]\\ i\neq t,t+1}} T_i^\perp,$$

where $T_{-1} := 0 =: T_{r+1}$.

REMARK.

For each $t \in [0, r+1]$ $\mathscr{V}_t$ is closed under extensions, direct summands, and images.

REMARK.

For each $t \in [0, r+1]$ the restriction of F to $\mathscr{V}_t[t]$ induces an equivalence with $\mathscr{U}_t$.

NOTATION.

For $t \in [0, r+1]$ let R_t be a chosen additive generator of the full subcategory of Ext-projective objects in $\mathscr{V}_t$.

NOTATION.

For $t \in [0, r+1]$ let L_t be a chosen additive generator of the full subcategory of Ext-injective objects in $\mathscr{V}_t$.

LEMMA.

Let $t \in [0, r+1]$ and $X \in \mathscr{V}_t$. If π is a minimal $\operatorname{add} R_t$-approximation, then π is surjective and $\operatorname{Ker}\pi \in \operatorname{add} R_t$.

LEMMA.

Let $t \in [0, r+1]$ and $X \in \mathscr{V}_t$. If α is a minimal $\operatorname{add} L_t$-approximation, then α is injective.

PROOF.

One easily checks that $\operatorname{Im}\alpha$ is Ext-injective in $\mathscr{V}_t$.

Proposition.

Let $t \in [0, r+1]$ and $X \in \mathscr{V}_t$. Then there exists a short exact sequence

$$0 \to R \to N \xrightarrow{f} X \to 0$$

such that $R \in \operatorname{add} R_t$, $N \in \operatorname{add}(R_t \oplus L_t)$, and f is an $\operatorname{add}(R_t \oplus L_t)$-approximation.

Proof.

Let $\pi : R' \to X$ be a minimal $\operatorname{add} R_t$-approximation and $\alpha : L' \to X$ a minimal $\operatorname{add} L_t$-approximation. Next, let

$$\begin{array}{ccc} R & \longrightarrow & L' \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ R' & \xrightarrow{\pi} & X \end{array}$$

be the pull-back diagram. Then we have an exact sequence

$$0 \to \operatorname{Ker} \pi \to R \to L' \to 0,$$

hence $R \in \mathscr{V}_t$. Moreover, β is injective, thus $R \in \operatorname{add} R_t$, and the claim follows.

Notation.

For $t \in [0, r+1]$ we put $Q_t := FR_t[t]$ and $K_t := FL_t[t]$.

Remark.

Let $t \in [0, r]$. If $X \in \mathscr{U}_t$ and $Y \in \mathscr{U}_{t+1}$, then any map $X \to Y$ factors through $\operatorname{add} Q_{t+1}$.

Theorem.

$\operatorname{rep.dim} \Lambda \leq 3$.

Proof.

Put $M := \bigoplus_{t \in [0,r+1]} (Q_t \oplus K_t)$. For an indecomposable Λ-module X we will find an exact sequence

$$\varepsilon : 0 \to M_1 \to M_0 \xrightarrow{f} X \to 0$$

such that $M_0, M_1 \in \operatorname{add} M$ and f is an $\operatorname{add} M$-approximation of X. Let $t \in [0, r+1]$ be such that $X \in \mathscr{U}_t$. Fix an indecomposable H-module Y such that $X = FY[t]$. We have an exact sequence

$$\eta : 0 \to R \to N \xrightarrow{g} X \to 0$$

such that $R \in \operatorname{add} R_t$, $N \in \operatorname{add}(R_t \oplus L_t)$, and g is an $\operatorname{add}(R_t \oplus L_t)$-approximation. We put $\varepsilon := F\eta$ and the claim follows.