MUTATIONS AND EQUIVALENCES

BASED ON THE TALK BY DONG YANG

The talk was based on a joint work with Bernhard Keller.

NOTATION.
For a finite quiver Q we denote by $k\hat{Q}$ the space consisting of the formal (possibly infinite) linear combinations of paths in Q.

NOTATION.
For a finite quiver Q we denote by $\mathscr{C}_Q$ the subspace of $k\hat{Q}$ consisting of the formal (possibly infinite) linear combinations of nontrivial cycles in Q. The elements of $\mathscr{C}_Q$ are called *potentials in* Q.

DEFINITION.
By *a quiver with potential* we mean a pair (Q, W) consisting of a finite quiver Q and a potential W in Q.

DEFINITION.
For an arrow ρ of a finite quiver Q we define *the cyclic derivative* $\partial_\rho : \mathscr{C}_Q \to k\hat{Q}$ by the following conditions:

(1) ∂_ρ commutes with infinite sums,
(2) if $c \in \mathscr{C}$ and $c = \rho_1 \cdots \rho_n$ for arrows $\rho_1, \ldots, \rho_n$ in Q, then

$$\partial_\rho c = \sum_{\substack{i \in [1,n] \\ \rho_i = \rho}} \rho_{i+1} \cdots \rho_n \rho_1 \cdots \rho_{i-1}.$$

DEFINITION.
For a finite quiver Q we define *the associated graded quiver* $\tilde{Q}$ as follows:

(1) the vertices of $\tilde{Q}$ coincide with the vertices of Q,
(2) $\tilde{Q}_1 := Q_1 \coprod Q_1^* \coprod \{t_i : i \to i \mid i \in Q_0\}$, where $Q_1^* := \{\rho^* : t\rho \to s\rho \mid \rho \in Q_1\}$,
(3) to each arrow of $\tilde{Q}$ we associate its degree as follows:

$$\deg \rho := 0, \ \rho \in Q_1, \quad \deg \rho^* := -1, \ \rho \in Q_1, \quad \deg t_i := -2, \ i \in Q_0.$$

DEFINITION.
By *the Ginzburg algebra* $\hat{\Gamma}(Q, W)$ a quiver with potential (Q, W) we mean a differential graded algebra $(k\hat{\tilde{Q}}, d)$ defined as follows:

Date: 03.07.2009.

(1) for $n \in \mathbb{Z}$, $k\hat{\tilde{Q}}_n$ consists of the formal (possibly infinite) linear combinations of paths in $\tilde{Q}$ of degree n,

(2) d commutes with infinite sums, satisfies the graded Leibnitz rule, i.e.

$$d(p \cdot q) = dp \cdot q + (-1)^{\deg p} \cdot p \cdot dq$$

for each homogeneous elements p and q of $k\hat{\tilde{Q}}$, and d is defined on the arrows by

$$d\rho := 0,\ \rho \in Q_1, \quad d\rho^* := \partial_\rho W,\ \rho \in Q_1,$$

and

$$dt_i := e_i\big(\sum_{\beta \in Q_1} \beta\beta^* - \beta^*\beta\big)e_i,\ i \in Q_0.$$

Definition.

By *the Jacobian algebra* $J(Q, W)$ of a quiver with potential (Q, W) we mean $H^0\hat{\Gamma}(Q, W)$.

Remark.

Given a quiver with potential (Q, W) and a vertex i of Q, such that there are no loops in Q and there are no 2-cycles at i, one defines a new quiver with potential called *the mutation of* (Q, W) *at* i and denoted $\tilde{\mu}_i(Q, W)$. There exists an injective quasi-isomorphism $\hat{\Gamma}(Q, W) \to \hat{\Gamma}(\tilde{\mu}_i^2(Q, W))$.

Theorem.

Let (Q, W) be a quiver with potential and i a vertex of Q. If there are no loops in Q and there are no 2-cycles at i, then

$$\operatorname{mod} J(\tilde{\mu}_i(Q, W))/S_i' \simeq \operatorname{mod} J(Q, W)/S_i,$$

where S_i and S_i' denote the corresponding simple modules.